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Spin(8) Gauge Field Theory

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A theoretical foundation for spin(8) gauge field theory is proposed to explain the numerical results announced in a previous paper.

1. SUMMARY

Spin(8) gauge field theory, as defined herein, has the following structure:

A Yang-Mills principal fibre bundle with base manifold S^4 and gauge group spin(8). The base manifold S^4 corresponds to space-time. The quaternionic structure of S^4 naturally corresponds to a $\{3, 3, 4, 3\}$ lattice structure of space-time that gives a naturally Lorentz-invariant lattice gauge theory.

The gauge group spin(8) decomposes at the Weyl group level into spin(5), $SU(3)$, spin(4), and the maximal torus $U(1)^4$.

The spin(5) component corresponds to a gauge theory of de Sitter gravitation with a cosmological term as described by MacDowell and Mansouri (1977).

The $SU(3)$ component corresponds to a gauge theory of the color force.

The spin(4) $= SU(2) \times SU(2)$ component corresponds to a gauge theory of the weak force with a geometric form of spontaneous symmetry breaking from spin(4) to $SU(2)$.

The $U(1)^4$ component corresponds to the four components of the photon in the path integral formulation of quantum electrodynamics.

For spinor matter fields, there is an associated bundle to the principal bundle that is related to the spinor representation of spin(8). One eight-dimensional half-spinor space corresponds to the first-generation fermion-particles as given in Table IV of Section 6. The mirror image eight-dimensional half-spinor space corresponds similarly to the first-generation fermion antiparticles.

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In spin(8) gauge field theory the speed of light, Planck's constant, and the electron mass are given. One can then calculate the quantities given in Table V of Section 18.

As will be seen, spin(8) gauge field theory is at present incomplete, but enough has been done so far that it seems very likely that the structure outlined above can be completed to produce a theory of physics that is in substantial accord with present experimental results.

Note on notation. I have tried to use standard notation throughout this paper, but since the subject matter is not standard, some nonstandard usages may appear. Examples include: using the symbol "×" for both the Cartesian product and a twitsted product; using the term "decomposition" when the whole object is not the direct sum of orthogonal factors; and referring to a Lie group G or Lie algebra g generally when in fact only a specific representation is being used. I hope that the meaning of such nonstandard usages is made clear by the context.

Further, references are not intended to be to the primary origin of the concept cited, but are instead to the reference I found most useful.

2. INTRODUCTION

In 1971, Armand Wyler wrote a paper in which he purported to calculate the fine structure constant to be $\alpha = 1/137.03608$ and the proton-electron mass ratio to be $m_p/m_e = 6\pi^5 = 1836.118$ from the volumes of homogeneous symmetric spaces (Wyler, 1971). Although the numerical values were close to experimental data, the physical reasons he gave for using the particular volumes he chose were not clear. Freeman Dyson invited Wyler to the Institute for Advanced Study in Princeton for a year to see if Wyler could develop good physical reasons. However, Wyler was primarily a mathematician and did not produce a convincing physical basis for his numerical calculations. With no clear physical basis, Wyler's results were dismissed by many physicists, such as those writing letters in the November 1971 issue of *Physics Today*, as mere unproductive numerology.

As far as I know, no further work was done on Wyler's results. It seemed to me that even though Wyler did not come up with a good physical basis, there might be one. To look for one, I started by trying to study generalizations of complex manifolds to quaternionic manifolds that have the structure of space-time.

In 1965, Wolf wrote a paper in which he classified the four-dimensional Riemannian symmetric spaces with quaternionic structure (Wolf, 1965). There are just four equivalence classes, with the following representatives:

$$T^4 = U(1)^4$$

$$S^2 \times S^2 = SU(2)/U(1) \times SU(2)/U(1)$$
$$CP^2 = SU(3)/S(U(2) \times U(1))$$
$$S^4 = \text{spin}(5)/\text{spin}(4)$$

Although Wolf's paper was pure mathematics with no attempt at physical application, it seemed to me that the occurrence of the gauge group of electromagnetism $U(1)$, the gauge group of the weak force $SU(2)$, the gauge group of the color force $SU(3)$, and the gauge group of de Sitter gravitation spin(5) might be physically significant.

Another indication of the possible physical significance of quaternionic structure was a paper of Finkelstein et al. (1963) in which they used quaternionic structure to construct a geometric spontaneous symmetry-breaking mechansim producing two charged massive vector bosons and one massless neutral photon.

I then started trying to construct a gauge field theory with a four-dimensional base manifold having quaternionic structure and a gauge group that would include $U(1)^4$, $SU(2) \times SU(2) = \text{spin}(4)$, $SU(3)$, and spin(5). Such a gauge group should have dimension at least $4+6+8+10=28$. If $U(1)^4$ is taken to be part of a maximal torus, the rank of the gauge group should be at least 4.

As spin(8) has rank 4 and dimension 28, it is a natural candidate. However, it does not even include $U(1) \times SU(2) \times SU(3)$ as a subgroup. To have $U(1)^4$, spin(4), $SU(3)$, and spin(5) included in it, spin(8) must be decomposed at the Weyl group level rather than decomposed into subgroups.

As noted by Günaydin and Gürsey (1973) and Georgi (1982), elements of spin(8) can be represented as triples of Pauli matrices. Triples of Pauli matrices act on triples of spinors. Triples of spinors correspond naturally to the classification of first-generation leptons and quarks by Harari (1979), Shupe (1979), and Adler (1980).

Günaydin and Gürsey (1973) showed that the triples of spinors are equivalent to octonions.

By applying Wyler's method to structure, I was able to calculate a number of particle masses and force strength constants (Smith, 1985). All were roughly consistent with currently accepted experimental results, except the truth quark mass $m_t \approx 130$ GeV. CERN has announced that $m_t \approx 45$ GeV (Rubbia, 1984), but I think that the phenomena observed by CERN at 45 GeV are weak force phenomena that are poorly explained by the standard $SU(2) \times U(1)$ model. I further think that current CP-violation experimental results (Wojcicki, 1985) are consistent with $m_t \approx 130$ GeV and the Kobayashi–Maskawa parameters calculated herein from spin(8) gauge field theory, and that the CERN value of $m_t \approx 45$ GeV is not consistent with those experimental results.

As of the summer of 1985, CERN has been unable to confirm its identification of the truth quark in the 45-GeV events, as the UA1 experimenters have found a lot of events clustering about the $W^{\pm}$ mass and the UA2 experimenters have not found anything convincing (Miller, 1985). I think the clustering of UA1 events near the $W^{\pm}$ mass indicates that the events observed are nonstandard weak force phenomena.

Since the mass of the t-quark and the Kobayashi-Maskawa parameters related to the t-quark are predictions of spin(8) gauge field theory that are experimentally determinable and can be used to confirm or refute spin(8) gauge field theory, I think that it is appropriate to call the t-quark the truth quark.

3. PREGEOMETRY AND CLIFFORD ALGEBRA

It is natural to look for more and more elementary foundations for a physical theory, with a goal of describing wider ranges of phenomena using fewer and simpler basic concepts. For instance, Misner et al. (1959) conclude their classic book *Gravitation* with a discussion of the possibility of deriving the laws of physics from a pregeometry based on the calculus of propositions.

Here, standard set theory is taken to be the elementary foundation from which physical theory should be constructed. From set theory, a rough sketch is given of the construction of pregeometry. Crude rules are set out for deriving spin(8) gauge field theory from pregeometry.

Begin with the empty set $\varnothing$, the operation { } called brace, the operation & called union, and the standard rules

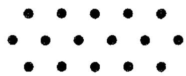
$$\text{if } s \text{ is a set, so is } (s)$$
$$\{s\&s\} = \{s\}$$
$$\text{if } s_1, \ldots, s_n \text{ are sets, so is } \{s_1\& \cdots \&s_n\}$$

The rank of a set is defined to be the maximal number of bracing operations used on $\varnothing$ within the set in the construction of the set.

The number of sets, including the null set, of rank at most k is denoted by $n(k)$ and given by $n(0) = 1$ and the recurrence relation $n(k) = 2^{n(k-1)}$.

The sets of rank at most k have a natural 1-1 correspondence with the subsimplexes of a topological simplex of $n(k-1)-1$ dimensions. To see this, note that an m-dimensional simplex has 2^{m+1} subsimplexes, with ${}_{m+1}C_j$ subsimplexes of dimension $j-1$, where the null set is given dimension -1. Consider the situation depicted in Table I. Rank 5 set theory has enough sets to define numbers counting up to $2^{65,536}$, or to put it another way, to define real numbers on the unit interval up to the accuracy of $2^{-65,536}$. As there may be about 10^{80} particles in the observed universe, rank 5 set theory is the lowest rank set theory that has the potential to describe our universe.

Table I.

$n(0)=1$	∅
$n(1)=2$	∅ {∅}
$n(2)=4$	∅ {{∅}} {∅&{∅}} {∅}
$n(3)=16$	
$n=(4)=65{,}536$	
$n(5)=2^{65{,}536}$	

Rank 5 set theory contains the empty set ∅ and everything of the form {. . . X. . .}, where . . . X. . . is a combination of the five fundamental objects of rank less than 5: ∅, {∅}, {{∅}}, {{{∅}}}, and {{{{∅}}}}. Of those five objects, the empty set ∅ is distinguished by the fact that it does not involve the bracing operation { }.

The five fundamental objects naturally form an oriented 4-simplex, called a tetrad: {∅&{∅}&{{∅}}&{{{∅}}}&{{{{∅}}}}}. It has five vertices, one of which is the distinguished empty set ∅. If ∅ is taken to be the origin, then it can be described by four vectors from the origin to each of the other four vertices. Each of the four vectors can be considered to be of unit length with sign either plus or minus. As rank 5 set theory contains numbers up to $2^{65,536}$, the length of the vectors can be defined and a four-dimensional vector space can be constructed. The tetrad can then be seen to be equivalent to the pseudoscalar of the $2^4=16$-dimensional Clifford algebra of $\mathbb{R}^4$. It is the highest dimensional Clifford algebra pseudoscalar in rank 5 set theory.

The fundamental physical significance of the tetrad has been studied by Finkelstein and Rodriguez (1982, 1984) and by Sirag (1981), whose works have been influential in developing this pregeometry. Keller (1984) has done work beginning with the pseudoscalar γ_5 of the Dirac Clifford algebra of $\mathbb{R}^{3,1}$. The finite group of the permutations and orientation-preserving sign changes of the four vectors of the tetrad is $S_4 \times Z_2^3$, the Weyl group of D_4, the Lie algebra of the simply connected Lie group spin(8).

As will be seen later, the gauge bosons, space-time, and spinor matter fields of spin(8) gauge field theory naturally correspond to the exceptional Lie group F_4, and the three generations of fermions naturally correspond to the exception Lie groups E_6, E_7, and E_8, so it is probable that spin(8) gauge field theory is truly fundamental. It is also probable that use of higher rank set theory will give nothing new, as there are no higher dimensional exceptional Lie groups and as it is unlikely that one needs to count higher than $2^{65,536}$.

Denote the 2^n-dimensional Clifford algebra of $\mathbb{R}^n$ by Cn, and its subspace of grade r by Cn_r, whose elements are called multivectors of grade r, or r-vectors (Hestenes and Sobczyk, 1984). The bivectors Cn_2 of Cn are closed under the commutator product, so that they form a Lie algebra. The bivector algebra, by exponentation and mapping to the universal covering group, gives spin(n).

From the Clifford algebra point of view, $S_4 \times Z_2^3$ can be thought of as the orientiation-preserving symmetry group of the four-dimensional oriented pseudoscalar $C4_4$ of $C4$, which has $1+4+6+4+1=2^4=16$ dimensions. The orientation-preserving symmetry group $S_4 \times Z_2^3$ corresponds to the permutations and the orientation-preserving changes of sign of the four vectors of $C4_1$. That finite symmetry group is the Weyl group of D_4. It has $4!\,2^{4-1}=4!\,2^3=192$ elements. It induces a finite symmetry group of the four-dimensional vector space $C4_1$.

Consider the Weyl group of D_4 as a finite reflection group in the four-dimensional vector space $C4_1$ with basis $\{e_1, \ldots, e_4\}$. It is the group of reflections in hyperplanes perpendicular to the $2(4-1)4=24$ points $\{\pm e_i \pm e_j \,|\, 1 \leq i < j \leq 4\}$, which points are called root vectors. The hyperplane perpendicular to the point $e_i - e_j$ permutes the vectors e_i and e_j, and the $\pm$ signs account for the orientation-preserving changes of sign. Note that in the case of D_4 the root vectors form a regular polytope, the 24-cell.

The 24 root vectors plus the four vectors of $C4_1$ form a 28-dimensional Lie algebra that is isomorphic to the $8(8-1)/2!=28$-dimensional bivector algebra $C8_2$, which is the Lie algebra of spin(8). The four vectors of $C4_1$ are the Cartan Clifford subalgebra of the D_4 Lie algebra of spin(8), the bivector algebra of the Clifford algebra $C8$. The algebra $C8$ has $2^8=256$ dimensions:

$$1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1 \qquad C8$$

The vector part of $C8$ is eight-dimensional, indicating that spin(8) acts naturally on an eight-dimensional vector space base manifold.

The expanded spinor representation of $C8$, and of the bivector algebra spin(8), has dimension $1+8+28+56+70+56+28+8+1=256=16\times 16$, and can be written as 16×16 matrices. It decomposes into even and odd parts that can each be written as 16×8 matrices, corresponding to the $1+28+70+28+1=128$-dimensional even part of $C8$ and the $8+56+56+8=128$-dimensional odd part of $C8$.

The (usual) spinor representation of $C8$ and spin(8) is the dimension of a minimal ideal of the 16×16 matrices, and can be written as a 16×1 column matrix. The 16-dimensional 16×1 spinor representation decomposes into two eight-dimensional 8×1 irreducible half-spinor representations corresponding to the even and odd parts of $C8$.

The half-spinor representations should correspond to the fermion spinor matter fields in spin(8) gauge field theory.

The eight-dimensional vector space base manifold in spin(8) gauge field theory should be reduced by a geometric Higgs mechanism described in Section 10 to a four-dimensional space-time base manifold. To see how that might work, consider that the vector space of the $C4$ Clifford algebra corresponds to the Cartan subalgebra of the bivector spin(8) Lie algebra of $C8$:

1	8	28	56	70	56	28	8	1	$C8$
1		4		6		4		1	Cartan $C4$

The four-dimensional space-time spin(8) base manifold should be the part of the eight-dimensional vector space $C8_1$ that is defined as follows: begin with the four-dimensional Cartan subalgebra of the spin(8) bivectors $C8_2$, which is defined by the four-dimensional vector space $C4_1$; then note that the Clifford product in $C4$ of the four vectors of $C4_1$ gives the $C4$ pseudoscalar $C4_4$; then note that the pseudoscalar Cn_n in a Clifford algebra Cn can be considered to be an n-dimensional volume element of the vector space Cn_1; then note that the pseudoscalar $C4_4$ defines a four-dimensional subspace in the $C8$ pseudoscalar $C8_8$; and then see that the corresponding four-dimensional subspace of the vector space $C8_1$ should correspond to the physically realistic space-time base manifold that remains after the action of the geometric Higgs mechanism in spin(8) gauge field theory. Note that this method identifies physical space-time with the four-dimensional vector space of the $C4$ Clifford algebra, which in turn can be naturally identified with the four standard Dirac γ_μ vectors of the $C(3,1)$ Clifford algebra.

The Cartan Clifford subalgebra $C4$ decomposes into even and odd irreducible components, of dimension $1+6+1=2^3=8$ and $4+4=2^3=8$. They are the expanded 4×2 half-spinor spaces for the $C4$ Clifford algebra, and correspond to the ordinary 8×1 half-spinor spaces of the spin(8) $C8$ Clifford algebra. The ordinary 2×1 half-spinor spaces for the $C4$ Clifford algebra, which correspond to the ordinary half-spinors of the Dirac equation, are subspaces of the ordinary spin(8) half-spinor spaces.

In summary, set theory gives the $C4$ Clifford algebra, whose symmetries generate the $C8$ Clifford algebra, which gives: a spin(8) gauge group; an eight-dimensional base manifold that can be reduced to a four-dimensional space-time; and two mirror image eight-dimensional half-spinor spaces corresponding to the first-generation fermion particles and antiparticles.

The underlying $C4$ Clifford algebra has a spin(4) bivector algebra that corresponds to the standard physical Lorentz group, and also has spin(4) half-spinors that correspond to the half-spinors of the Dirac equation.

The use of $C4$ pseudoscalar symmetry to generate the Weyl group of the spin(8) bivector algebra of the $C8$ Clifford algebra is similar to Keller's use of the γ_5 Dirac pseudoscalar symmetry (Keller, 1984).

It might be that spin(8) gauge field theory can be given a more unified structure by considering larger groups.

The Clifford fibration spin(8) $\rightarrow$ spin(9) $\rightarrow S^8$ shows that spin(9) is the twisted product of the gauge group spin(8) and the unreduced base manifold S^8 (Atiyah et al., 1964; Porteous, 1981; Grossman et al., 1984). Therefore spin(9) may include all the structure of spin(8), gauge field theory except that of the spinor matter field within itself, by identifying the eight infinitesimal generator root vectors of spin(9) not of spin(8) as corresponding to the unreduced space-time base manifold.

It might be possible also to include the spinor matter field, if the 16-dimensional full spinor space of spin(8) gauge field theory could be identified with the 16-dimensional Cayley projective plane $\mathbb{O}P^2$. The fibration spin(9) $\rightarrow F_4 \rightarrow \mathbb{O}P^2$ shows that F_4 is the twisted product of spin(9) and $\mathbb{O}P^2$ [see Besse (1978), where F_4 is described as the automorphism group of the exceptional 27-dimensional Jordan algebra $M_3(\mathbb{O})$ of 3×3 Hermitian matrices of octonions with the product $A\circ B=(1/2)(AB+BA)$].

Then $F_4=\text{spin}(8)\times S^8\times \mathbb{O}P^2$ could include all the structure of spin(8) gauge field theory, including the fermion spinor lepton and quark particles and antiparticles, which would correspond to the 16 infinitesimal generator root vectors of F_4 not of spin(9).

Note that F_4 may be the "core" of the E-series of exceptional Lie algebras, in the sense that G_2 is the "core of spin(7), spin(8), spin(9), and F_4, due to the fibration $G_2 \rightarrow \text{spin}(7) \rightarrow S^7$ (Porteous, 1981), with spin(7) being the twisted product $G_2\times S^7$ of the automorphism group of octonions $\mathbb{O}$ and the imaginary octonions, so that;

$$\begin{aligned}
\text{spin}(7) &= G_2\times S^7\\
\text{spin}(8) &= G_2\times S^7\times S^7\\
\text{spin}(9) &= G_2\times S^7\times S^7\times S^8\\
F_4 &= G_2\times S^7\times S^7\times S^8\times \mathbb{O}P^2
\end{aligned}$$

Denote by $M_3(\mathbb{O})_0$ the 26-dimensional subspace of $M_3(\mathbb{O})$ having trace 0. Then, following the Freudenthal-Tits magic square described in McCrimmon (1978), it may be possible to show that

$$E_6=F_4\times M_3(\mathbb{O})_0$$

$$E_7 = F_4 \times S^3 \times M_3(\mathbb{O})_0 \times M_3(\mathbb{O})_0 \times M_3(\mathbb{O})_0$$
$$E_8 = F_4 \times G_2 \times M_3(\mathbb{O})_0 \times M_3(\mathbb{O})_0 \times M_3(\mathbb{O})_0 \times M_3(\mathbb{O})_0 \times M_3(\mathbb{O})_0 \times M_3(\mathbb{O})_0 \times M_3(\mathbb{O})_0$$

It may be that the above scheme is related to the following irreducible symmetric spaces;

$E_6/\text{spin}(10) \times S^1$ of dimension 32 and rank 2
$E_7/\text{spin}(12) \times S^3$ of dimension 64 and rank 4
$E_8/\text{spin}(16)$ of dimension 128 and rank 8

As $M_3(\mathbb{O})$ naturally coordinatizes $\mathbb{O}P^2$ (McCrimmon, 1978) and $F_4/\text{spin}(9) = \mathbb{O}P^2$ of dimension 16 and rank 1, it may be that

$$E_6/\text{spin}(10) \times S^1 = \mathbb{O}P^2 \times \mathbb{O}P^2$$
$$E_7/\text{spin}(12) \times S^3 = \mathbb{O}P^2 \times \mathbb{O}P^2 \times \mathbb{O}P^2 \times \mathbb{O}P^2$$
$$E_8/\text{spin}(16) = \mathbb{O}P^2 \times \mathbb{O}P^2 \times \mathbb{O}P^2 \times \mathbb{O}P^2 \times \mathbb{O}P^2 \times \mathbb{O}P^2 \times \mathbb{O}P^2 \times \mathbb{O}P^2$$

If $F_4 = \text{spin}(8) \times S^8 \times \mathbb{O}P^2$ describes action of the gauge group spin(8) acting within the space-time S^8 on the fermions of $\mathbb{O}P^2$, what about the fermion generation structure?

The first-generation fermions may correspond to $M_3(\mathbb{O})_0$, in which $\mathbb{O}P^2$ is naturally imbedded. What is needed is a map from the $\mathbb{O}P^2$ in F_4 to the $M_3(\mathbb{O})_0$. Such a map naturally occurs in the structure of $E_6 = F_4 \times M_3(\mathbb{O})_0$, being the map to the $M_3(\mathbb{O})_0$ in E_6 corresponding to the one imaginary generator of the complex numbers. E_6 may describe the physics of the first-generation fermions.

The second-generation fermions may correspond to $M_3(\mathbb{O})_0 \times M_3(\mathbb{O})_0$. A map from the $\mathbb{O}P^2$ in F_4 to $M_3(\mathbb{O})_0 \times M_3(\mathbb{O})_0$ occurs naturally in

$$E_7 = F_4 \times S^3 \times M_3(\mathbb{O})_0 \times M_3(\mathbb{O})_0 \times M_3(\mathbb{O})_0$$

by taking the map to the $M_3(\mathbb{O})_0 \times M_3(\mathbb{O})_0$ in E_7 corresponding to the two imaginary generators of the quaternions. E_7 may describe the physics of the second-generation fermions.

The third-generation fermions may correspond to $M_3(\mathbb{O})_0 \times M_3(\mathbb{O})_0 \times M_3(\mathbb{O})_0$. A map from the $\mathbb{O}P^2$ in F_4 to $M_3(\mathbb{O})_0 \times M_3(\mathbb{O})_0 \times M_3(\mathbb{O})_0$ occurs naturally in

$$E_8 = F_4 \times G_2 \times M_3(\mathbb{O})_0 \times M_3(\mathbb{O})_0 \times M_3(\mathbb{O})_0 \times M_3(\mathbb{O})_0 \times M_3(\mathbb{O})_0 \times M_3(\mathbb{O})_0 \times M_3(\mathbb{O})_0$$

by taking the map to the $M_3(\mathbb{O})_0 \times M_3(\mathbb{O})_0 \times M_3(\mathbb{O})_0$ in E_8 corresponding to the three imaginary generators of the octonions. E_8 may describe the physics of the second-generation fermions.

E_6, E_7, and E_8 may correspond to the first-, second-, and third-generation fermions. There should be no higher generations.

4. WEYL GROUP DECOMPOSITION OF SPIN(8) AND GAUGE BOSONS

The Weyl group of spin(8) determines how the gauge bosons interact with the elementary fermions, so it would be useful to have a correspondence between the elements of the Weyl group and the elements of the Lie algebra. Spin(8) is a rank 4 group, so that its Weyl group acts as Euclidean 4-space reflections about hyperplanes perpendicular to the 24 Lie algebra root vector elements. There is no 1-1 correspondence between the 192 elements of the Weyl group and the 24 Lie algebra root vectors. Such a 1-1 correspondence exists only for Lie groups of rank 2 or less. Therefore, it is useful to decompose spin(8) at the Weyl group level into Lie groups of rank no greater than 2.

$S_4 \times Z_2^3$ is the semidirect product of $S_2 \times Z_2^2$, S_3, $S_2 \times S_2$, and the identity. Consider that spin(8) naturally decomposes into (Table II): spin(5) = sp(2), the gauge group of de Sitter gravitation according to MacDowell and Mansouri (1977), with ten infinitesimal generators, each corresponding to a graviton, $SU(3)$, the color force group, with eight infinitesimal generators, each corresponding to a gluon; spin(4) = $SU(2) \times SU(2)$, which decomposes by spontaneous symmetry-breaking into the $SU(2)$ of the weak force, with three infinitesimal generators, each corresponding to a massive weak boson; and $U(1)^4$, the maximal torus, with four infinitesimal generators, each corresponding to one of the four polarizations of the photon described by Leighton (1959).

The Weyl group of spin(8) can be visualized geometrically by beginning with the unique four-dimensional, centrally symmetric, self-dual regular polytope called the 24-cell or (3, 4, 3) (Coxeter, 1973). The 24-cell has 24 vertices and 24 octahedral three-dimensional faces. It can be described in quaternionic coordinates $\{1, i, j, k\}$ by the vertices $(\pm 1 \pm i)/\sqrt{2}$, $(\pm 1 \pm j)/\sqrt{2}$, $(\pm 1 \pm k)/\sqrt{2}$, $(\pm i \pm j)/\sqrt{2}$, $(\pm i \pm k)/\sqrt{2}$, and $(\pm j \pm k)/\sqrt{2}$.

To visualize a 24-cell, project it into $\mathbb{R}^3$ with two antipodal octahedra centered about the origin, one inside the other. Then the other 22 octahedra

Table II.

Weyl group $S_4 \times Z_2^3$	Number of elements 192	Group spin(8)	Dimension of group 28
$S_2 \times Z_2^2$	8	spin(5) = sp(2)	10
S_3	6	$SU(3)$	8
$S_2 \times S_2$	4	spin(4) = $SU(2) \times SU(2)$	6
Identity, maximal torus of spin(8)		$U(1)^4$	4

lie between the inner and outer central octahedra. There are six octahedra that share a vertex with the inner octahedron and another vertex with the outer octahedron, eight octahedra that share a face with the inner octahedron, and eight octahedra that share a face with the outer octahedron.

As a 24-cell is self-dual, the 24 points at the centers of the octahedra also form a 24-cell. The quaternionic coordinates of the dual 24-cell are ± 1, $\pm i$, $\pm j$, $\pm k$ and $(\pm 1 \pm i \pm j \pm k)/2$, which are the unit integral quaternions (Coxeter, 1973).

Call Aut(24) the group of linear transformations of $\mathbb{R}^4$ leaving the 24-cell invariant. Let v be a vertex of the 24-cell. Call r_v a reflection along the vertex v iff $r_v(v) = -v$ and the fixed points of r_v constitute a hyperplane in $\mathbb{R}^4$. Call such a hyperplane a Weyl hyperplane. Then the Weyl group of spin(8) is the subgroup of Aut(24) generated by the r_v for all vertices v of the 24-cell.

Weyl chambers are defined as the connected components of $\mathbb{R}^4$ into which it is divded by all the hyperplanes of fixed points of all the r_v. The Weyl group of spin(8) can be seen to be the group of reflections that permute the Weyl chambers in a simply transitive way (Bourbaki, 1968).

For spin(8), there are 12 Weyl hyperplanes and 192 Weyl chambers. There are eight Weyl chambers for each of the 24 octahedral faces of the 24-cell. The 24 octahedral faces can be seen as four loops of six octahedral faces each (Coxeter, 1973).

The Weyl group of $\text{spin}(9) = S^8 \times \text{spin}(8)$ is $S_2 \times S_4 \times Z_2^3 = S_2 \times$ Weyl group of spin(8). The 36 root vector vertices of spin(9) are the spin(8) 24-cell and ± 1, $\pm i$, $\pm j$, and $\pm k$ of the dual 24-cell. The four vertices $+1$, $+i$, $+j$, and $+k$ correspond to space-time, with $+1$ corresponding to time. The vertices -1, $-i$, $-j$, and $-k$ correspond to the four dimensions of S^8 that are reduced by the geometric Higgs mechanism for reducing S^8 to S^4.

The Weyl group of

$$F_4 = \mathbb{O}P^2 \times \text{spin}(9) = \mathbb{O}P^2 \times S^8 \times \text{spin}(8)$$

is $S_3 \times S_4 \times Z_2^3 = S_3 \times$ Weyl group of $\text{spin}(8) = Z_3 \times$ Weyl group of spin(9). The 48 root vector vertices of F_4 are those of spin(9) and $(\pm 1 \pm i \pm j \pm k)/2$ of the dual 24-cell or both the 24-cell and its dual 24-cell. The eight vertices $(+1+i+j+k)/2$, $(+1+i+j-k)/2$, $(+1+i-j+k)/2$, $(+1-i+j+k)/2$, $(+1-i-j+k)/2$, $(+1-i+j-k)/2$, $(+1+i-j-k)/2$, and $(+1-i-j-k)/2$ correspond to the electron, red up quark, blue up quark, green up quark, red down quark, blue down quark, green down quark, and neutrino. The eight vertices $(-1 \pm i \pm j \pm k)/2$ correspond to the fermion antiparticles.

The four polarizations of the photon $[U(1)^4]$ correspond to the common Cartan subalgebra of F_4, spin(9), and spin(8).

Note that the vertices of the 24-cell of spin(8) can be related to the vertices of the dual 24-cell by quaternionic right-multiplication by $(1+j)/\sqrt{2}$, which takes $1\to(1+j)/\sqrt{2}$, $i\to(i+k)/\sqrt{2}$, $j\to(-1+j)/\sqrt{2}$, and $k\to(-i+k)/\sqrt{2}$.

Then the 24 infinitesimal generators of spin(5), $SU(3)$, and spin(4) correspond to vertices of the 24-cell of spin(8) as follows:

1. Eight root vectors of spin(5): $(-1\pm i)/\sqrt{2}$, $(-1\pm k)/\sqrt{2}$, $(+j\pm i)/\sqrt{2}$, and $(+j\pm k)/\sqrt{2}$, which are related to the eight fermion antiparticles $(-1\pm i\pm j\pm k)/2$.
2. Six root vectors of $SU(3)$: $(+1-i)/\sqrt{2}$, $(+1\pm k)/\sqrt{2}$, $(-j+i)/\sqrt{2}$, and $(-j\pm k)/\sqrt{2}$, which are related to the six fermion quarks $(+1-i+j-k)/2$, $(+1-i+j+k)/2$, $(+1+i+j-k)/2$, $(+1+i-j+k)/2$, $(+1-i-j+k)/2$, and $(+1+i-j-k)/2$.
3. Four Cartan subalgebra elements, two of spin(5) and two of $SU(3)$: $(+1-j)/\sqrt{2}$, $(+i-k)/\sqrt{2}$, $(+1+j)/\sqrt{2}$, and $(+i+k)/\sqrt{2}$, which are related to $+1$, $+i$, $+j$, and $+k$ of space-time.
4. Two root vectors of weak $SU(2)$ of spin(4): $(+1+i)/\sqrt{2}$ and $(-j-i)/\sqrt{2}$, which are related to the fermion electron $(+1+i+j+k)/2$ and neutrino $(+1-i-j-k)/2$.
5. One Cartan subalgebra element of weak $SU(2)$ of spin(4): $(-1+j)/\sqrt{2}$, which is related to the S^8 Higgs element -1 and produces the W^0.
6. Three Higgs $SU(2)$ elements of spin(4): $(-i+k)/\sqrt{2}$, $(-1-j)/\sqrt{2}$, and $(-i-k)/\sqrt{2}$, which are related to the S^8 Higgs elements $-i$, $-j$, and $-k$.

As the eight root vectors of spin(5) are related to all eight fermion antiparticles, the eight charged gravitons can carry both electric and color charges.

As the six root vectors of $SU(3)$ are related only to the six quarks, the six charged gluons can carry only color charge.

As the two root vectors of weak $SU(2)$ are related only to the electron and the neutrino, the charged W^+ and W^- can carry only electric charge.

5. GEOMETRY OF SPIN(8)

Grossman et al. (1984) have shown that the last Hopf map $S^7\to S^{15}\to S^8$ produces a principal fibre bundle $(P, \pi, S^8, \text{spin}(8))$ that corresponds to the Yang-Mills equations with gauge group spin(8), base manifold S^8, and topological charge 1.

The 8-sphere $S^8=\text{spin}(9)/\text{spin}(8)$ is a compact manifold with local symmetry group spin(8). The Yang-Mills base manifold S^8 should correspond to space-time. As will be seen in Section 10, S^8 can be reduced to S^4 to get a space-time of the right dimensionality. Physical space-time is usually considered to be a noncompact manifold such as $\mathbb{R}^4$ rather than the compact manifold S^4. However, Yang–Mills over a four-dimensional base manifold is conformally invariant, so that every solution of Yang–Mills over S^4 pulls back to a finite-energy solution over $\mathbb{R}^4$ by the conformal identification $\mathbb{R}^4 \approx S^4 - \{\infty\}$ (Uhlenbeck, 1985). Note that both $\mathbb{R}^n$ and S^n have spin(n) as local symmetry group. The compact base manifold S^4 has a finite volume and other geometrical properties that are useful in the calculations of spin(8) gauge field theory. Similar use of compact manifolds rather than the projectively related noncompact manifolds was made by Wyler (1971) and discussed by Gilmore (1972).

As described in Besse (1978), the Grassmannian manifold of oriented 2-planes in $\mathbb{R}^{10}$, denoted by $G^+_{2,10} = \text{spin}(10)/\text{spin}(8) \times S^1$, where $S^1 = \text{spin}(2)$, is the manifold of oriented geodesics of S^9, denoted by $C_{2\pi}S^9$, because the geodesics of S^9 are of length 2π. As $\mathbb{R}P^9$ is covered twofold by S^9, $G^+_{2,10}$ is also the manifold or oriented geodesics of $\mathbb{R}P^9$, denoted by $C_\pi \mathbb{R}P^9$, because the geodesics of $\mathbb{R}P^9$ are of length π.

$G^+_{2,10}$ is isomorphic to a bounded complex homogeneous domain of type IV_8, denoted by D^8. If D^8 is taken to be equivalent to $C_{2\pi}S^9$, the Silov boundary of D^8 is $Q^8 \approx S^7 \times S^1$. If $D^8_\pm$ is taken to be equivalent to $C_\pi \mathbb{R}P^9$, the Silov boundary of $D^8_\pm$ is $Q^8_\pm \approx S^7 \times \mathbb{R}P^1$ (Hua, 1963).

$Q^8_\pm$ is parallelizable and compact, and is used in Section 6 as the manifold for half-spinor spaces for spin(8) gauge field theory.

We have already seen that the Yang–Mills gauge group spin(8) corresponds at the Weyl group level to the physical gauge groups spin(5) of de Sitter gravitation, $SU(3)$ of the color force, spin(4), which is reduced to $SU(2)$ with three massive vector bosons by geometric spontaneous symmetry breaking, and $U(1)^4$ of electromagnetism.

To see the relationship of the Weyl group decomposition of spin(8) to the group structure of spin(8), follow Porteous (1981) and recall that $\text{spin}(8) = S^7 \times S^7 \times G_2$:

1. Use the Hopf fibration $S^3 \to S^7 \to S^4$,

$$\text{spin}(8) \to S^4 \times S^3 \times S^4 \times S^3 \times G_2$$

2. Use the map $G_2 \to S^6 \times SU(3)$,

$$\text{spin}(8) \to SU(3) \times (S^4 \times S^6) \times (S^3 \times S^3) \times S^4$$

3. Identify $SU(3)$ with the color force (Günaydin, 1976).
4. Identify $S^4 \times S^6$ with $\text{spin}(5) = \text{sp}(2)$, shown by MacDowell and Mansouri (1977) to be the gauge group of de Sitter gravitation. S^4

should correspond to the translations of $\mathbb{R}^4$ of which S^4 is a compactification. S^6, whose almost complex structure is based on pairs of imaginary quaternions (Kobayashi and Nomizu, 1963, 1969), should correspond to the Lorentz transformations of $\text{spin}(4) = S^3 \times S^3$,

$$\text{spin}(8) \to SU(3) \times \text{spin}(5) \times (S^3 \times S^3) \times S^4$$

5. Identify $S^3 \times S^3 = \text{spin}(4)$ with the weak force and the Higgs mechanism,

$$\text{spin}(8) \to SU(3) \times \text{spin}(5) \times \text{spin}(4) \times S^4$$

6. Identify S^4 with $U(1)^4$ by identifying S^4 with the translations of $\mathbb{R}^4$ of which S^4 is a compactification.

$$\text{spin}(8) \to \text{spin}(5) \times SU(3) \times \text{spin}(4) \times U((1)^4$$

The structure of $\text{spin}(8) = S^7 \times S^7 \times G_2$ is such that the Lie algebra of spin(8) is generated by the infinitesimal generators of S^7 and their Lie algebra commutators $[S^7, S^7]$ (Günaydin and Gürsey, 1973). The decomposition of spin(8) can be written in terms of the basis for spin(8) as the bivector algebra of the $2^8 = 256$-dimensional Clifford algebra of $\mathbb{R}^8$, writing 16×16 matrices as direct products of two Pauli matrices ρ_i and τ_j and one Dirac matrix γ_μ, which basis was developed by Chisholm and Farwell (1984) (Table III):

$$\rho_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \rho_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\tau_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\gamma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\gamma_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma_3 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

Table III.

Scalar			
$\rho_0\tau_0\gamma_0$			I_{16}
Vectors			
$\rho_1\tau_3\gamma_1$	$\rho_1\tau_3\gamma_5$		Γ_1 Γ_5
$\rho_1\tau_3\gamma_2$	$\rho_0\tau_1\gamma_0 i$		Γ_2 Γ_6
$\rho_1\tau_3\gamma_3$	$\rho_0\tau_2\gamma_0 i$		Γ_3 Γ_7
$\rho_1\tau_3\gamma_4$	$\rho_2\tau_3\gamma_0$		Γ_4 Γ_8
Bivectors			
$\rho_0\tau_0\gamma_{12}$	$\rho_0\tau_0\gamma_{51}$	de Sitter gravitation—spin(5)	$\Gamma_1\Gamma_2$ $\Gamma_1\Gamma_5$
$\rho_0\tau_0\gamma_{23}$	$\rho_0\tau_0\gamma_{52}$		$\Gamma_2\Gamma_3$ $\Gamma_2\Gamma_5$
$\rho_0\tau_0\gamma_{34}$	$\rho_0\tau_0\gamma_{53}$		$\Gamma_3\Gamma_4$ $\Gamma_3\Gamma_5$
$\rho_0\tau_0\gamma_{14}$	$\rho_0\tau_0\gamma_{54}$		$\Gamma_4\Gamma_1$ $\Gamma_4\Gamma_5$
$\rho_0\tau_0\gamma_{13}$			$\Gamma_1\Gamma_3$
$\rho_0\tau_0\gamma_{24}$			$\Gamma_2\Gamma_4$
$\rho_1\tau_2\gamma_1 i$	$\rho_1\tau_1\gamma_1 i$	Color $SU(3)$	$\Gamma_1\Gamma_6$ $\Gamma_1\Gamma_7$
$\rho_1\tau_2\gamma_2 i$	$\rho_1\tau_1\gamma_2 i$		$\Gamma_2\Gamma_6$ $\Gamma_2\Gamma_7$
$\rho_1\tau_2\gamma_3 i$	$\rho_1\tau_1\gamma_3 i$		$\Gamma_3\Gamma_6$ $\Gamma_3\Gamma_7$
$\rho_1\tau_2\gamma_4 i$	$\rho_1\tau_1\gamma_4 i$		$\Gamma_4\Gamma_6$ $\Gamma_4\Gamma_7$
$\rho_2\tau_2\gamma_0 i$		Higgs $SU(2)$ part of spin(4)	$\Gamma_8\Gamma_6$
$\rho_2\tau_1\gamma_0 i$			$\Gamma_8\Gamma_7$
$\rho_0\tau_3\gamma_0$			$\Gamma_6\Gamma_7$
$\rho_1\tau_2\gamma_5 i$		Massive weak boson $SU(2)$ part of spin(4)	$\Gamma_5\Gamma_6$
$\rho_1\tau_1\gamma_5 i$			$\Gamma_5\Gamma_7$
$\rho_3\tau_0\gamma_5$			$\Gamma_5\Gamma_8$
$\rho_3\tau_0\gamma_1$		$U(1)^4$ electromagnetism	$\Gamma_1\Gamma_8$
$\rho_3\tau_0\gamma_2$			$\Gamma_2\Gamma_8$
$\rho_3\tau_0\gamma_3$			$\Gamma_3\Gamma_8$
$\rho_3\tau_0\gamma_4$			$\Gamma_4\Gamma_8$

$$I_{16} = \rho_0\tau_0\gamma_0$$

$$\Gamma_1 = \rho_1\tau_3\gamma_1, \qquad \Gamma_2 = \rho_1\tau_3\gamma_2, \qquad \Gamma_3 = \rho_1\tau_3\gamma_3$$

$$\Gamma_4 = \rho_1\tau_3\gamma_4, \qquad \Gamma_5 = \rho_1\tau_3\gamma_5, \qquad \Gamma_6 = \rho_0\gamma_1\gamma_0 i$$

$$\Gamma_7 = \rho_0\tau_2\gamma_0 i, \qquad \Gamma_8 = \rho_2\tau_3\gamma_0, \qquad \Gamma_9 = \rho_3\tau_3\gamma_0$$

The decomposition of the bivectors of the Chisolm-Farwell representation of spin(8) is more particularly described as follows

De Sitter Gravitation—Spin(5). As $\rho_0 = \tau_0 = 1$, the Chisolm-Farwell bivectors are just the six bivectors $\{\gamma_{\mu\nu}\}$ of spin(4), the covering group of the Lorentz group, and the four pseudovectors $\{\gamma_{5\mu}\}$ could correspond to the translations in four-dimensional space-time.

Color SU(3). $\{\rho_1\tau_2\gamma_1 i, \rho_1\tau_2\gamma_2 i, \rho_1\tau_2\gamma_3 i, \rho_1\tau_2\gamma_4 i\}$ can be identified with $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ by projection into the hyperspace orthogonal to $\rho_1\tau_2$ and ignoring the factor i. By the same process, $\{\rho_1\tau_1\gamma_1 i, \rho_1\tau_1\gamma_2 i, \rho_1\tau_1\gamma_3 i, \rho_1\tau_1\gamma_4 i\}$ can be identified with $\{\tau_1\gamma_1, \tau_1\gamma_2, \tau_1\gamma_3, \tau_1\gamma_4\}$. By identifying $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ with $\mathbb{H}$ and $\{\tau_1\gamma_1, \tau_1\gamma_2, \tau_1\gamma_3, \tau_1\gamma_4\}$ with $\tau_1\mathbb{H}$, and then identifying $\mathbb{H}\oplus\tau_1\mathbb{H}$ with the octonions $\mathbb{O}$, the eight bivectors $\{\rho_1\tau_2\gamma_1 i, \rho_1\tau_2\gamma_2 i, \rho_1\tau_2\gamma_3 i, \rho_1\tau_2\gamma_4 i, \rho_1\tau_1\gamma_1 i, \rho_1\tau_1\gamma_2 i, \rho_1\tau_1\gamma_3 i, \rho_1\tau_1\gamma_4 i\}$ are identified with $\mathbb{O}$.

The adjoint representation of $SU(n)$ is a subalgebra of spin(n^2-1) (Cahn, 1984). Therefore the eight-dimensional adjoint representation of $SU(3)$ is a subalgebra of spin(8). In the eight-dimensional representation of spin(8), considered as actions of spin(8), on $\mathbb{O}$, every action of spin(8) can be represented by octonion multiplication (Günaydin and Gürsey, 1973, Appendix C). Therefore the eight-dimensional adjoint representation of $SU(3)$ should be representable by the octonions $\mathbb{O}$.

Higgs SU(2) *Part of Spin*(4). $\{\rho_2\tau_1 i, \rho_2\tau_2 i, \tau_3\}$ can be identified with one of the $SU(2)$'s in spin (4) by projection into the hyperspace orthogonal to ρ_2 and ignoring the factor i. By a similar process, the vectors $\{\tau_1 i, \tau_2 i, \rho_2\tau_3\}$ will also correspond to $\{\tau_1, \tau_2, \tau_3\}$ and can be paired with the bivectors $\{\rho_2\tau_1 i, \rho_2\tau_2 i, \tau_3\}$ as described in Section 10.

Massive Weak Boson SU(2) *Part of Spin*(4). $\{\rho_1\tau_2\gamma_5 i, \rho_1\tau_1\gamma_5 i, \rho_3\tau_0\gamma_5\}$ can be identified with the other $SU(2)$ in spin(4) by projection into the hyperspace orthogonal to ρ_1, ρ_3, and γ_5; ignoring the factor i; and identifying τ_0 with τ_3 because $\tau_0 = {}^*\tau_3$ in the Pauli algebra. The τ_1 and τ_2 elements would correspond to the charged weak bosons. The $\tau_0 = {}^*\tau_3$ element would correspond to the neutral weak boson. By a similar process, the vector $\{\rho_1\tau_3\gamma_5\}$ will also correspond to $\tau_0 = {}^*\tau_3$ and can be paired with the bivector $\{\rho_3\tau_0\gamma_5\}$ as described in Section 10.

Electromagnetism—U$(1)^4$. $\{\rho_3\gamma_\mu\}$ can be identified with translations in four-dimensional space-time by projection into the hyperspace orthogonal to ρ_3.

The Chisolm–Farwell representation is such that de Sitter spin(5) gravitation sits inside spin(8) in a substantially trivial manner. Explicit calculation of commutator structure constants for the 16×16 bivector matrices should give interesting topological information about how the other components $SU(3)$, spin(4), and $U(1)^4$ fit together.

The

$$\text{spin}(8) = G_2\times S^7\times S^7 = SU(3)\times S^6\times S^4\times S^3\times S^1\times S^2\times S^4$$

decomposition is such that $SU(3)$ sits inside spin(8) in a substantially trivial manner. Further analysis of that decomposition might be useful, particularly with respect to $SU(3)$.

6. FERMION MATTER FIELDS AND $Q^8 = Q^8_+ \oplus Q^8_-$

The principal fibre bundle (P, π, S^8, spin(8)) gives a physically realistic gauge group spin(8) and a compact base manifold S^8 that will be shown to be reducible to S^4 for a physically realistic space-time, so that we have gauge forces acting in space-time. However, the principal fibre bundle alone does not give any matter fields upon which the gauge forces act. To get the matter fields, it is necessary to construct an associated bundle whose fibres correspond to the spinor representation of spin(8).

The Clifford algebra of $\mathbb{R}^8$ has dimension

$$2^8 = 16 \times 16 = 256 = 1 + 8 + 28 + 56 + 70 + 56 + 28 + 8 + 1$$

The 28-dimensional bivector space corresponds to the adjoint representation of spin(8). The eight-dimensional vector space corresponds to the vector representation. To get the spinor representation, imbed spin(8) in a 16×16 matrix representation of the whole Clifford algebra. A minimal ideal upon which spin(8) acts is then a 16×16 matrix with only the first column nonzero, or equivalently, a 16-dimensional column vector. The 16-dimensional column vector can be taken to be the spinor representation space Σ^8. It is reducible to two eight-dimensional mirror image irreducible half-spinor spaces Σ^8_+ and Σ^8_- that respectively lie in the even and odd subspaces of the spin(8) Clifford algebra. $\Sigma^8 = \Sigma^8_+ \oplus \Sigma^8_-$ is noncompact and has infinite volume. For the purposes of spin(8) gauge field theory calculations it is necessary to use a corresponding compact spinor space as done by Wyler (1971) and discussed by Gilmore (1972).

As Σ^8_+ and Σ^8_- are mirror images of each other, it is natural to consider Σ^8 to be a twofold covering space of $\Sigma^8_\pm \approx \Sigma^8_+ \approx \Sigma^8_-$.

$\Sigma^8_\pm$ is an eight-dimensional space that is isomorphic to $\mathbb{R}^8$ and has octonionic structure. $\Sigma^8_\pm$ is parallelizable in the sense that there exist eight independent tangent vector fields, which are linearly independent at every point. Parallelizability is physically important because, after choice of any point of $\Sigma^8_\pm$ as origin, each of the eight independent vector fields then corresponds to a basis vector of $\Sigma^8_\pm$, and each basis vector can then be taken to correspond to one of the eight leptons and quarks of the first-generation particles (for Σ^8_+) or antiparticles (for Σ^8_-). The octonionic structure classifies the leptons and quarks according to the scheme of Harari (1979), Shupe (1979), and Adler (1980).

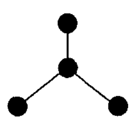
The compact space that corresponds to $\Sigma^8_\pm$ should also be eight-dimensional and parallelizable with octonionic structure. The most obvious candidate, $\mathbb{R}P^8$ covered by S^8, is not suitable because it is not parallelizable.

However, $\mathbb{R}P^1$ covered by S^1 is parallelizable and S^7 is also parallelizable. S^7 corresponds to the imaginary octonions and $\mathbb{R}P^1$ can be taken to correspond to the real axis, so that $S^7 \times \mathbb{R}P^1$ covered by $S^7 \times S^1$ is the corresponding compact structure to $\Sigma^8_\pm$ covered by Σ^8.

Denote by $Q^8_+ \approx S^7 \times \mathbb{R}P^1$ the compact space corresponding to Σ^8_+, and denote by $Q^8_- \approx S^7 \times \mathbb{R}P^1$ the compact space corresponding to Σ^8_-. Denote by $Q^8 \approx Q^8_+ \oplus Q^8_- \approx S^7 \times S^1$ the twofold covering compact space corresponding to the twofold covering spinor space $\Sigma^8 \approx \Sigma^8_+ \oplus \Sigma^8_-$.

$Q^8_\pm = S^7 \times \mathbb{R}P^1$ is the Silov boundary of an irreducible symmetric bounded domain of type IV_8, denoted by $D^8_\pm \approx \text{spin}(10)/\text{spin}(8) \times U(1)$ (Hua, 1963). $D^8_\pm$ has local symmetries of spin(8) and $U(1)$ (Gilmore, 1974; Gürsey and Tze, 1980), so that the spin(8) gauge group acts naturally on $D^8_\pm$ as well as its Silov boundary $Q^8_\pm$. The $U(1)$ local symmetry corresponds to the complex structure of $D^8_\pm$, which has eight complex dimensions.

Note that for spin($2k$) the mirror image left-handed and right-handed elementary (half spinor representations have dimension 2^{k-1}. For $\text{spin}(2 \times 4) = \text{spin}(8)$, $2^{4-1} = 2^3 = 8$, so the elementary spinor representations of spin(8) each have the same dimension as the vector space on which spin(8) acts. Further, the $1+3+3+1$ graded structure of each elementary spinor representation is compatible with the octonionic structure of the vector space on which spin(8) acts. Identification of the pseudoscalar of an elementary spinor representation with an imaginary octonion, say $O7$, of the vector representation gives an isomorphism between the spinor and vector representations of spin(8). Such an isomorphism is unique to spin(8), whose Dynkin diagram is



Consider the left-handed (half) spinor space Q^8_+. As one of the (half) spinor representation spaces of spin(8), it is isomorphic to both the other (half) spinor representation space Q^8_- and the vector representation space of spin(8). Since the vector representation space of spin(8) is just the octonions $\mathbb{O}$, the spin(8) isomorphisms show the equivalence of (half) spinor representations of spin(8), which are naturally written as triples of Pauli matrices, with the vector representation of spin(8), which is naturally written in terms of octonions (Günaydin and Gürsey, 1973).

Note that octonion multiplication can be defined in terms of a basis $\{1, O1, O2, O3, O4, O5, O6, O7\}$ as follows:

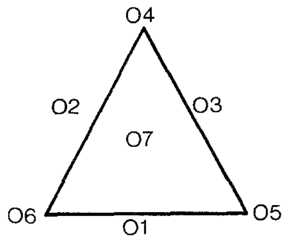


Fig. 1.

Begin with a triangle considered as the null set, three vertices, three edges, and the whole triangle (Figure 1). Identify, as shown in Figure 1, 1 with the null set; $O4$, $O5$, and $O6$ with the vertices; $O1$, $O2$, and $O3$ with the edges; and $O7$ with the whole triangle. Then the octonionic product with the null set is the identity; the octonionic product of two vertices is the edge between them, with a minus sign if the order of the vertices is clockwise and a plus sign if counterclockwise; the octonionic product of two edges is the third edge, with a plus sign if the order of the edges is clockwise and a minus sign if counterclockwise; the octonionic product of an edge and an adjacent vertex is the other adjacent vertex, with a plus sign if clockwise and a minus sign if counterclockwise; and the octonionic product of a vertex times the whole triangle is the opposite edge.

Note that it is natural to identify the electrically neutral massless neutrino with the null set, the charge $-1/3$ down quarks with the vertices, the charge $+2/3$ up quarks with the edges, and the charge -1 electron with the whole triangle.

Q_+^8 can be given a basis as a (half) spinor representation space in terms of standard spinors U and D, or, equivalently, in terms of octonions {1, $O1$, $O2$, $O3$, $O4$, $O5$, $O6$, $O7$}, as shown in Table IV.

Table IV.

Fermion	Triple of spinors	Octonion
Electron	$U\otimes U\otimes U$	$O7$
Green up quark	$D\otimes U\otimes U$	$O3$
Blue up quark	$U\otimes D\otimes U$	$O2$
Red up quark	$U\otimes U\otimes D$	$O1$
Green down quark	$U\otimes D\otimes D$	$O6$
Blue down quark	$D\otimes U\otimes D$	$O5$
Red down quark	$D\otimes D\otimes U$	$O4$
Neutrino	$D\otimes D\otimes D$	1

Similarly, the right-handed mirror image eight-dimensional spinor space Q^8_- corresponds to the eight elementary first-generation antiparticles: positron; antigreen, antiblue, and antired up antiquarks; antigreen, antiblue, and antired down antiquarks; and antineutrino.

The massless neutrino and antineutrino travel at the speed of light and cannot change their observed helicity, so that all neutrinos should be observed to be left-handed and all antineutrinos to be right-handed. The other fermions are massive and travel at less than the speed of light, so that their observed helicities can be of either type, depending on their velocities relative to observers.

The second generation of fermions should correspond to action of spin(8) on pairs of octonions (Smith, 1985). The octonion multiplication product is used to relate pairs of octonions to octonions (Günaydin and Gürsey, 1973; Hasiewicz and Kwasniewski, 1985).

The third generation of fermions should correspond to the action of spin(8) on triples of octonions (Smith, 1985). The octonion triple product of Whitehead (1962) is used to relate triples of octonions to octonions.

As there are no similar products of n-tuples of octonions for $n > 3$ (Eckmann, 1968; Whitehead, 1962), the nth-order half-spinor representations of spin(8) should not correspond to physically observable particles for $n > 3$. There should be no leptons or quarks beyond the third generation.

The same conclusion is reached if the leptons and quarks of the first, second, and third generations are identified, as in Section 3, with E_6, E_7, and E_8 respectively, based on $F_4 = \text{spin}(8) \times S^8 \times \mathbb{O}P^2$ describing the action of the gauge group spin(8) acting within the space-time S^8 on the fermions of $\mathbb{O}P^2$, and;

$$E_6 = F_4 \times M_3(\mathbb{O})_0$$

with the $M_3(\mathbb{O})_0$ corresponding to the one imaginary generator of the complex numbers;

$$E_7 = F_4 \times S^3 \times M_3(\mathbb{O})_0 \times M_3(\mathbb{O})_0 \times M_3(\mathbb{O})_0$$

with two of the $M_3(\mathbb{O})_0$ corresponding to the two imaginary generators of the quaternions; and

$$E_8 = F_4 \times G_2 \times M_3(\mathbb{O})_0 \times M_3(\mathbb{O})_0 \times M_3(\mathbb{O})_0 \times M_3(\mathbb{O})_0 \times M_3(\mathbb{O})_0 \times M_3(\mathbb{O})_0 \times M_3(\mathbb{O})_0$$

with three of the $M_3(\mathbb{O})_0$ corresponding to the three imaginary generators of the octonions.

7. ACTION OF GAUGE BOSONS ON ELEMENTARY FERMIONS

The photons are Abelian elements of $U(1)$, which is identical with its Cartan subalgebra. The Cartan subalgebra corresponds to the identity of the Weyl group. Therefore the action of photons on elementary fermions is described by the identity automorphism of octonions. Photons do not change the nature of elementary fermions, or, to put it another way, photons carry neither electric nor color charge.

The massive weak bosons W^+ and W^- are the two Lie algebra root vectors of $SU(2)$ and W^0 is the Cartan subalgebra. W^+ and W^- correspond to the Weyl group S_2, so their action on the elementary fermions is described by the octonion automorphism

$$\{O7, O1, O2, O3\} \leftrightarrow \{1, O4, O5, O6\}$$

W^+ and W^- carry one unit of electric charge, so that they can interchange neutrinos (1) with electrons ($O7$) or interchange down quarks ($O4$, $O5$, or $O6$) with up quarks ($O1$, $O2$, or $O3$).

The six gluons gl-1, ..., gl-6 are the six Lie algebra root vectors of $SU(3)$, and the two gluons gl-7 and gl-8 are the Cartan subalgebra. gl-1, ..., gl-6 correspond to the Weyl group S_3, so their action on the elementary fermions is described by the octonion automorphisms

$$\begin{aligned} \{O1, O4\} &\leftrightarrow \{O2, O5\} \\ \{O2, O5\} &\leftrightarrow \{O3, O6\} \\ \{O3, O6\} &\leftrightarrow \{O1, O4\} \end{aligned}$$

gl-1, ..., gl-6 can carry only color charge.

The eight gravitons gr-1, ..., gr-8 are the eight Lie algebra root vectors of $Sp(2)$, and the two gravitons gr-9 and gr-10 are the Cargan subalgebra. gr-1, ..., gr-8 correspond to the Weyl group $S_2 \times Z_2^2$, so their action on the elementary fermions is described by the octonion automorphisms

$$\begin{aligned} \{O7, O1, O2, O3\} &\leftrightarrow \{1, O4, O5, O6\} \\ \{O7, O1, O2, O3\} &\leftrightarrow \{O6, O5, O4, 1\} \\ \{O7, O1, O2, O3\} &\leftrightarrow \{O5, O6, 1, O4\} \end{aligned}$$

Note that the composition of the three automorphisms gives $\{O7, O1, O2, O3\} \leftrightarrow \{O4, 1, O6, O5\}$, and the gravitons are the only gauge bosons that can change leptons into quarks and quarks into leptons. gr-1 and gr-2 can carry only electric charge, but gr-3, ..., gr-8 can carry both electric and color charge.

The following list summarizes the action of charged gauge bosons on the elementary fermion particles of the first generation. The action on antiparticles and higher generation particles can be deduced from the action on first-generation particles.

W^+: $\{O7, O1, O2, O3\} \to \{1, O4, O5, O6\}$
W^-: $\{1, O4, O5, O6\} \to \{O7, O1, O2, O3\}$

gl-1: $\{O1, O4\} \to \{O2, O5\}$
gl-2: $\{O1, O4\} \to \{O3, O6\}$
gl-3: $\{O2, O5\} \to \{O3, O6\}$
gl-4: $\{O2, O5\} \to \{O1, O4\}$
gl-5: $\{O3, O6\} \to \{O1, O4\}$
gl-6: $\{O3, O6\} \to \{O2, O5\}$

gr-1: $\{O7, O1, O2, O3\} \to \{1, O4, O5, O6\}$
gr-2: $\{1, O4, O5, O6\} \to \{O7, O1, O2, O3\}$
gr-3: $\{O7, O1, O2, O3\} \to \{O6, O5, O4, 1\}$
gr-4: $\{O6, O5, O4, 1\} \to \{O7, O1, O2, O3\}$
gr-5: $\{O7, O1, O2, O3\} \to \{O5, O6, 1, O4\}$
gr-6: $\{O5, O6, 1, 04\} \to \{O7, O1, O2, O3\}$
gr-7: $\{O7, O1, O2, O3\} \to \{O4, 1, O6, O5\}$
gr-8: $\{O4, 1, O6, O5\} \to \{O7, O1, O2, O3\}$

8. QUARK MASS

What is the ratio of the mass of a down quark to the mass of an electron, m_d/m_e? According to the Weyl group decomposition of spin(8), 18 infinitesimal generators correspond to photons, weak bosons, and gluons, and 10 correspond to gravitons. The m_d/m_e mass ratio should be made up of two parts: the ratio of "expansion factors" due to the photon, weak boson, and gluon interactions on the down quark and the electron; and the ratio of the numbers of gravitons that interact with the down quark and the electron.

Consider a red down quark, $O4$. By gluon interactions, $O4$ can be taken into $O5$ and $O6$. By weak boson interactions, it can be taken into $O1$, $O2$, and $O3$. By the octonionic product, $O4 \times O6 \times O5 = O7$ and $O4 \times O2 \times O6 = 1$. Therefore the red down quark (similarly, any down quark) can be "expanded" by a factor of the volume of $Q^8_+ = V(Q^8_+)$.

Consider an electron, $O7$. By photon, weak boson, and gluon interactions, $O7$ can only be taken into 1, the massless neutrino. By the octonion product, $O7$ and 1 can only be taken into the subspace of the octonions spanned by $O7$ and 1. Therefore the electron cannot be "expanded" at all. Its "expansion factor" is just 1.

The ratio of the down quark expansion factor to the electron expansion factor is just $V(Q^8_+)/1 = V(Q^8_+) = \pi^5/3$ (Hua, 1963).

The ten gravitons correspond to the ten infinitesimal generators of spin(5) = $Sp(2)$. Two of them are in the Cartan subalgebra. Six of them carry color charge, and may therefore be considered as corresponding to quarks. The remaining two carry no color charge, but may carry electric charge and so may be considered as corresponding to electrons. One takes the electron into itself, and the other can only take the first-generation electron into the massless electron neutrino. Therefore only the one graviton should correspond to the mass of the first-generation electron.

The graviton number ratio of the down quark to the first-generation electron is therefore $6/1 = 6$.

Therefore the ratio of the down quark constituent mass to the electron mass is $m_d/m_e = 6V(Q^8_+) = 2\pi^5 = 612.03937$. If the electron mass is $m_e = 0.5110034$ MeV, then the constituent mass of the down quark is $m_d = 312.7542$ MeV.

As the up quarks correspond to $O1$, $O2$, and $O3$, which are isomorphic to $O4$, $O5$, and $O6$ of the down quarks, the up quarks and down quarks have the same mass $m_u = m_d = 312.7542$ MeV.

9. FERMION MASSES—SECOND AND THIRD GENERATIONS

The second-generation fermion particles correspond to pairs of octonions. There are $8^2 = 64$ such pairs. The pair $(1, 1)$ corresponds to the μ-neutrino. The pairs $(1, O7)$, $(O7, 1)$, and $(O7, O7)$ correspond to the μ. Compare the symmetries of the μ pairs to the symmetries of the first-generation fermion particles. The pair $(O7, O7)$ should correspond to the $O7$ electron. The other two μ pairs have a symmetry group S_2, which is one-third the size of the color symmetry group S_3, which gives the up and down quarks their mass of 312.7542 MeV. Therefore the mass of the μ should be the sum of the electron mass and one-third of the up or down quark mass, and $m_\mu = 104.7624$ MeV.

Note that all pairs correspond to the μ and the μ-neutrino are colorless.

The strange quark corresponds to the nine pairs $(1, O4)$, $(1, O5)$, $(1, O6)$, $(O4, 1)$, $(O5, 1)$, $(O6, 1)$, $(O4, O4)$, $(O5, O5)$, and $(O6, O6)$. Its mass should come from two sources: the other two-thirds of the down quark mass that is not associated with the μ mass, or 208.5028 MeV; and the μ mass times the graviton factor. Unlike the first-generation situation, massive second- and third-generation leptons can be taken, by both of the colorless gravitons that may carry electric charge, into massive particles. Therefore the graviton factor for the second and third generations is $6/2 = 3$. Therefore the μ mass times the graviton factor 3 is 314.2872 MeV, and the strange quark constituent mass is $m_s = 522.7900$ MeV.

The red strange quark is defined as the three pairs $(1, O4)$, $(O4, 1)$, and $(O4, O4)$, because $O4$ is the red down quark.

The blue strange quarks correspond to the three pairs involving $O5$, and the green strange quarks correspond to the three pairs involving $O6$.

The charm quark corresponds to the other 51 pairs. Its mass should also come from two sources: the other two-thirds of the up quark mass that is not associated with the μ mass, or 208.5028 MeV; and 51/9 times the μ part of the strange quark mass, or 1780.9608 MeV. Therefore the charm quark constituent mass is $m_c = 1.9894636$ GeV.

The red charm quark is defined as the following 17 pairs: $(O1, 1)$, $(1, O1)$, and $(O1, O1)$, because $O1$ is the red up quark; $(O2, O3)$, $(O3, O2)$, $(O7, O4)$, and $(O4, O7)$, because the octonion product of the elements of each such pair is $\pm O1$, the red up quark; $(O3, O5)$, $(O5, O3)$, $(O6, O2)$, $(O2, O6)$, $(O6, O5)$, $(O5, O6)$, $(O7, O1)$, and $(O1, O7)$, because the octonion product of the elements of each such pair is $\pm O4$, the red down quark; and $(O1, O4)$ and $(O4, O1)$, because both elements of each such pair are red up or down quarks.

The blue and green charm quarks are defined similarly.

The third-generation fermion particles correspond to triples of octonions. There are $8^3 = 512$ such triples. The triple $(1, 1, 1)$ corresponds to the τ-neutrino. The other seven triples involving only 1 and $O7$ correspond to the tauon. The symmetry of the seven tauon triples is the same as the symmetry of the three down quarks, the three up quarks, and the electron, so the tauon mass should be the same as the sum of the masses of the first-generation massive fermion particles. Therefore the tauon mass is $m_{\text{tau}} = 1.877036$ GeV.

Note that all triples correspond to the τ and the τ-neutrino are colorless.

The bottom quark corresponds to 21 triples. The are triples of the form $(1, 1, O4)$, $(1, O4, 1)$, $(O4, 1, 1)$, $(O4, O4, 1)$, $(O4, 1, O4)$, $(1, O4, O4)$, and $(O4, O4, O4)$, and the similar triples for 1 and $O5$ and for 1 and $O6$. The bottom quark constituent mass should be the tauon mass times the graviton factor 3, or $m_b = 5.631108$ GeV.

The red bottom quark is defined as the seven triples $(1, 1, O4)$, $(1, O4, 1)$, $(O4, 1, 1)$, $(O4, O4, 1)$, $(O4, 1, O4)$, $(1, O4, O4)$, and $(O4, O4, O4)$, because $O4$ is the red down quark.

The blue bottom quarks correspond to the seven triples involving $O5$, and the green bottom quarks correspond to the seven triples involving $O6$.

The truth quark corresponds to the remaining 483 triples, so the constituent mass of the truth quark is 483/21 times the bottom quark mass, or $m_t = 129.51548$ GeV.

The red truth quark is defined as the following 161 triples:

$(1, 1, O1)$, $(1, O1, 1)$, $(O1, 1, 1)$, $(O1, O1, 1)$
$(O1, 1, O1)$, $(1, O1, O1)$, $(O1, O1, O1)$

because $O1$ is the red up quark;

$(O7, O2, O6)$, $(O7, O6, O2)$, $(O7, O1, O7)$
$(O7, O7, O1)$, $(O7, O3, O5)$, $(O7, O3, O5)$
$(O7, 1, O4)$, $(O7, O4, 1)$

$(O6, O4, O3)$, $(O6, O3, O4)$, $(O6, O6, O1)$
$(O6, O1, O6)$, $(O6, O2, O7)$, $(O6, O7, O2)$
$(O6, 1, O5)$, $(O6, O5, 1)$

$(O5, O4, O2)$, $(O5, O2, O4)$, $(O5, O5, O1)$
$(O5, O1, O5)$, $(O5, O3, O7)$, $(O5, O7, O3)$
$(O5, 1, O6)$, $(O5, O6, 1)$

$(O4, O6, O5)$, $(O4, O5, O6)$, $(O4, O4, O1)$
$(O4, O1, O4)$, $(O4, O2, O3)$, $(O4, O3\ \ O2)$
$(O4, 1, O7)$, $(O4, O7, 1)$

$(O3, O6, O4)$, $(O3, O4, O6)$, $(O3, O3, O1)$
$(O3, O1, O3)$, $(O3, O5, O7)$, $(O3, O7, O5)$
$(O3, 1, O2)$, $(O3, O2, 1)$

$(O2, O4, O5)$, $(O2, O5, O4)$, $(O2, O2, O1)$
$(O2, O1, O2)$, $(O2, O6, O7)$, $(O2, O7, O6)$
$(O2, 1, O3)$, $(O2, O3, 1)$

$(O1, O7, O7)$, $(O1, O6, O6)$, $(O1, O5, O5)$
$(O1, O4, O4)$, $(O1, O3, O3)$, $(O1, O2, O2)$

$(1, O7, O4)$, $(1, O4, O7)$, $(1, O6, O5)$
$(1, O5, O6)$, $(1, O3, O2)$, $(1, O2, O3)$

because the octonion triple products defined by $xyz = x((x^{-1}y)(x^{-1}z))$ (Eckmann, 1968; Whitehead, 1962) of the elements of each such triple is $\pm O1$, the red up quark;

$(O7, O5, O6)$, $(O7, O6, O5)$, $(O7, O4, O7)$
$(O7, O7, O4)$, $(O7, O3, O2)$, $(O7, O3, O2)$
$(O7, 1, O1)$, $(O7, O1, 1)$

$(O6, O4, O6)$, $(O6, O6, O4)$, $(O6, O3, O1)$
$(O6, O1, O3)$, $(O6, O5, O7)$, $(O6, O7, O5)$
$(O6, 1, O2)$, $(O6, O2, 1)$

$$(O5, O4, O5), (O5, O5, O4), (O5, O2, O1)$$
$$(O5, O1, O2), (O5, O6, O7), (O5, O7, O6)$$
$$(O5, 1, O3), (O5, O3, 1)$$
$$(O4, O7, O7), (O4, O6, O6), (O4, O5, O5)$$
$$(O4, O3, O3), (O4, O2, O2), (O4, O1, O1)$$
$$(O3, O6, O1), (O3, O1, O6), (O3, O3, O4)$$
$$(O3, O4, O3), (O3, O2, O7), (O3, O7, O2)$$
$$(O3, 1, O5), (O3, O5, 1)$$
$$(O2, O1, O5), (O2, O5, O1), (O2, O2, O4)$$
$$(O2, O4, O2), (O2, O3, O7), (O2, O7, O3)$$
$$(O2, 1, O6), (O2, O6, 1)$$
$$(O1, O6, O3), (O1, O3, O6), (O1, O5, O2)$$
$$(O1, O2, O5), (O1, O4, O1), (O1, O1, O4)$$
$$(O1, 1, O7), (O1, O7, 1)$$
$$(1, O7, O1), (1, O1, O7), (1, O6, O2)$$
$$(1, O2, O6), (1, O3, O5), (1, O5, O3)$$

because the octonion triple product defined by $xyz = x((x^{-1}y)(x^{-1}z))$ (Eckmann, 1968; Whitehead 1962) of the elements of each such triple is $\pm O4$, the red down quark;

$$(O7, O4, O4), (O7, O1, O1)$$
$$(O4, O6, O5), (O4, O5, O6), (O4, O2, O3)$$
$$(O4, O3, O2), (O4, O4, O1), (O4, O1, O4)$$
$$(O4, 1, O1), (O4, O1, 1)$$
$$(O3, O2, O4), (O3, O4, O2)$$
$$(O2, O3, O4), (O2, O4, O3)$$
$$(O1, O1, O7), (O1, O7, O1)$$
$$(O1, 1, O4), (O1, O4, 1)$$
$$(1, O4, O1), (1, O1, O4)$$

because although the octonion triple product defined by $xyz = x((x^{-1}y)(x^{-1}z))$ (Eckmann, 1968; Whitehead, 1962) of the elements of each such triple is colorless $\pm O7$, the dominant elements are $O4$ or $O1$; and

$$(O7, O4, O1), (O7, O1, O4)$$
$$(O6, O1, O5), (O6, O5, O1)$$
$$(O5, O1, O6), (O5, O6, O1)$$
$$(O4, O1, O7), (O4, O7, O1)$$

$$(O1, O6, O5), (O1, O6, O5), (O1, O2, O3)$$
$$(O1, O2, O3), (O1, O4, O7), (O1, O7, O4)$$

because although the octonion triple product defined by $xyz = x((x^{-1}y)(x^{-1}z))$ (Eckmann, 1968; Whitehead, 1962) of the elements of each such triple is colorless ± 1, the dominant elements are $O4$ or $O1$.

The blue and green truth quarks are defined similarly.

Are there any more quarks and leptons than those in the first three generations? If so, an extension of these calculations (Smith, 1985) predicts that the masses of the heavy leptons of the fourth, fifth, and sixth generations should be under about 8 GeV and that the masses of the down-type quarks of those generations should be under about 23 GeV. However, in June 1984 it was reported that DESY has looked at electron–positron collisions up to 45 GeV and found no such new heavy leptons or quark–antiquark pairs [*Scientific American* **250**(6), 80 (1984)]. As such experiments should have found any new heavy leptons under 8 GeV and any new quarks under 22.5 GeV, there seem to be no fourth, fifth, or sixth generations of fermions and probably no generations other than the first three.

Spin(8) gauge field theory indicates that this should be the case. To get the correspondence between the pairs of octonions of the second-order half-spinor representation of spin(8) and the second-generation leptons and quarks, the octonion multiplication product was used to relate pairs of octonions to octonions. To get the correspondence between the triples of octonions and the third-generation leptons and quarks, the octonion triple product of Whitehead (1962) was used to relate triples of octonions to octonions. As there are no similar products of n-tuples of octonions for $n > 3$ (Eckmann, 1968; Whithead, 1962), the nth-order half-spinor representations of spin(8) should not correspond to physically observable particles for $n > 3$. There should be no leptons or quarks beyond the third generation.

The same conclusion is reached if the leptons and quarks of the first, second, and third generations are identified, as in Section 3, with E_6, E_7, and E_8, respectively, based on $F_4 = \text{spin}(8) \times S^8 \times \mathbb{O}P^2$ describing the action of the gauge group spin(8) acting within the space-time S^8 on the fermions of $\mathbb{O}P^2$, and

$$E_6 = F_4 \times M_3(\mathbb{O})_0$$

with the $M_3(\mathbb{O})_0$ corresponding to the one imaginary generator of the complex numbers;

$$E_7 = F_4 \times S^3 \times M_3(\mathbb{O})_0 \times M_3(\mathbb{O})_0 \times M_3(\mathbb{O})_0$$

with two of the $M_3(\mathbb{O})_0$ corresponding to the two imaginary generators of the quaternions; and

$$E_8 = F_4 \times G_2 \times M_3(\mathbb{O})_0 \times M_3(\mathbb{O})_0 \times M_3(\mathbb{O})_0 \times M_3(\mathbb{O})_0 \times M_3(\mathbb{O})_0 \times M_3(\mathbb{O})_0 \times M_3(\mathbb{O})_0$$

with three of the $M_3(\mathbb{O})_0$ corresponding to the three imaginary generators of the octonions.

10. GEOMETRIC HIGGS REDUCTION OF S^8 BASE MANIFOLD TO S^4

Mayer (1981a, b) has described a geometric Higgs mechanism that he calls "fiber-flipping" that should be useful for reducing the spin(8) Yang–Mills gauge field theory over S^8 to a Yang–Mills–Higgs theory over S^4. Therefore, we should get only a physically realistic S^4 space-time base manifold, but also massive $SU(2)$ weak vector bosons.

To use the notation of Mayer (1981a, b), let the gauge group $G = \text{spin}(8)$ with Yang–Mills base manifold $E = S^8$ of dimension $4+k$, with $k=4$. The desired four-dimensional base manifold is $M = S^4$. Using the geometric decomposition

$$\text{spin}(8) = S^7 \times S^7 \times G_2 = S^3 \times S^1 \times S^2 \times S^4 \times S^4 \times S^6 \times SU(3)$$

described in Section 5, let

$$H = S^2 \times S^4 \times S^4 \times G_2 = S^2 \times S^4 \times S^4 \times S^6 \times SU(3)$$

Although H is not a subgroup of G as contemplated by Mayer (1981), G/H is well-defined. H is 24-dimensional, and G/H has dimension $k = 4 = \dim E - \dim M$. Further, the Lie algebra of $G = \text{spin}(8)$ can be split such that $G = H + G/H$.

$G/H = S^3 \times S^1$. In terms of the Chisolm–Farwell basis for the bivectors of the 16×16 spin(8) Clifford algebra, G/H should correspond to

$$\begin{aligned} S^3 &= \{\rho_2\tau_2\gamma_0 i, \rho_2\tau_1\gamma_0 i, \rho_0\tau_3\gamma_0\} \\ &= \{\Gamma_8\Gamma_6, \Gamma_8, \Gamma_7, \Gamma_6, \Gamma_7\} \\ &= \text{Higgs } SU(2) \text{ part of spin(4) component of spin(8)} \end{aligned}$$

plus

$$\begin{aligned} S^1 &= \{\rho_3\tau_0\gamma_5\} \\ &= \{\Gamma_5\Gamma_8\} \\ &= \text{neutral part of massive weak boson } SU(2) \\ &\quad \text{part of spin(4) component of spin(8)} \end{aligned}$$

The physical base manifold corresponds to space-time, and is usually taken to be $\mathbb{R}^4$. However, Yang–Mills structure over the compact base manifold S^4 is equivalent by the conformal identification $\mathbb{R}^4 = S^4 - \{\infty\}$ (Uhlenbeck, 1985). Prior to dimensional reduction to S^4, the base manifold of spin(8) gauge field theory is S^8. For spin(8) Gauge field theory, the compact base manifold S^8 is considered to be equivalent to $\mathbb{R}^8 = S^8 - \{\infty\}$. Further, since

$$S^8 = S^7 \times (\mathbb{R}P^1 - \{\infty\}) \cup \{0\} \cup \{\infty\}$$

where $\mathbb{R}P^1$ is the semicircle covered twofold by S^1, the base manifold E is also considered to be equivalent to $S^7 \times \mathbb{R}P^1$, so that $E = S^8 \approx S^7 \times \mathbb{R}P^1 = S^4 \times S^3 \times \mathbb{R}P^1$.

$E/M = S^3 \times \mathbb{R}P^1$. In terms of the Chisolm–Farwell basis for the vectors of the 16×16 spin(8) Clifford algebra, E/M should correspond to

$$S^3 = \{\rho_0 \tau_1 \gamma_0 i, \rho_0 \tau_2 \gamma_0 i, \rho_2 \tau_3 \gamma_0\} = \{\Gamma_6, \Gamma_7, \Gamma_8\}$$

plus $\mathbb{R}P^1 = \{\rho_1 \tau_3 \gamma_5\} = \{\Gamma_5\}$.

$M = S^4$ should correspond to

$$\{\rho_1 \tau_3 \gamma_1, \rho_1 \tau_3 \gamma_2, \rho_1 \tau_3 \gamma_3, \rho_1 t_3 \gamma_4\} = \{\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4\}$$

There should be a splitting $E = M + E/M$ corresponding to the splitting $G = H + G/H$.

Note that reduction from E to M as base manifold by integrating G/H over E/M leaves a remainder term in the gauge group of $\mathbb{R}P^1$ because the S^1 term in G/H is a double cover of the $\mathbb{R}P^1$ term in E/M. Physically, that means that the resulting gauge group over base manifold $M = S^4$ is not H, but is

$$\begin{aligned} \mathbb{R}P^1 \times H &= \mathbb{R}P^1 \times S^2 \times S^4 \times S^4 \times S^6 \times SU(3) \\ &\approx S^3 \times S^4 \times (S^4 \times S^6) \times SU(3) \end{aligned}$$

$SU(3)$ corresponds to color $SU(3)$ (Günaydin, 1976).

The S^4 with Chisolm–Farwell basis

$$\{\rho_0 \tau_0 \gamma_{51}, \rho_0 \tau_0 \gamma_{52}, \rho_0 \tau_0 \gamma_{53}, \rho_0 \gamma_{54}\} = \{\Gamma_1\Gamma_5, \Gamma_2\Gamma_5, \Gamma_3, \Gamma_5, \Gamma_4\Gamma_5\}$$

corresponds to four-dimensional translations of the $\mathbb{R}^4$ of which S^4 is a compactification. S^6 has almost complex structure based on pairs of imaginary quaternions (Kobayashi and Nomizu, 1963, 1969), which structure is locally like $S^3 \times S^3 = SU(2) \times SU(2) = \text{spin}(4)$, the Lie algebra of the covering group of the Lorentz transformations. Therefore $S^4 \times S^6$ corresponds to de Sitter gravitation.

The other S^4 with Chisolm–Farwell basis

$$\{\rho_3 \tau_0 \gamma_1, \rho_3 \tau_0 \gamma_2, \rho_3 \tau_0 \gamma_3, \rho_3 \tau_0 \gamma_4\} = \{\Gamma_1\Gamma_8, \Gamma_2\Gamma_8, \Gamma_3\Gamma_8, \Gamma_4\Gamma_8\}$$

corresponds to electromagnetism.

Of the $\mathbb{R}P^1 \times S^2 \approx S^3$, the S^2 with Chisolm–Farwell basis $\{\rho_1\tau_2\gamma_0 i, \rho_1\tau_1\gamma_0 i\} = \{\Gamma_5\Gamma_6, \Gamma_5\Gamma_7\}$ corresponds to the two charged massive weak $SU(2)$ bosons and the $\mathbb{R}P^1$ with Chisolm–Farwell basis $\{\rho_3\tau_0\gamma_5\} = \{\Gamma_5\Gamma_8\}$ corresponds to the neutral massive weak $SU(2)$ boson. Half of the S^1 $\{\Gamma_5\Gamma_8\}$ was "eaten" by the $\mathbb{R}P^1$ in E/M with Chisolm–Farwell basis $\{\Gamma_5\}$, leaving the other half of the S^1 that double covers $\mathbb{R}P^1$.

The Higgs $SU(2)$ bivectors with Chisolm–Farwell basis $\{\Gamma_8\Gamma_6, \Gamma_8\Gamma_7, \Gamma_6\Gamma_7\}$ were "eaten" by combining with the S^3 vectors in E/M with Chisolm–Farwell basis $\{\Gamma_6, \Gamma_7, \Gamma_8\}$.

The geometric Higgs mechanism works as follows:

Let A_8 be a $G = \text{spin}(8)$ 1-form on $E = S^8 = \mathbb{R}^8 \cup \{\infty\}$, and F_8 a $G = \text{spin}(8)$ 2-form on E. Then A_8 is in the space of the eight vectors of the spin(8) Clifford algebra, and F_8 is in the space of the 28 bivectors of the spin(8) Clifford algebra.

Consider the Yang–Mills action $\int_{S^8} F_8 \wedge {}^*F_8 + \langle \varphi, \partial_8 \varphi \rangle$, in which the $F_8 \wedge {}^*F_8$ term is the curvature terms and the $\langle \varphi, \partial_8 \varphi \rangle$ term is the spinor matter term.

The objective is to reduce the Yang–Mills structure over S^8 with vector potential, or connection, A_8, bivector curvature F_8, and Dirac operator ∂_8 acting on spinor spaces to a Yang–Mills–Higgs structure over S^4 with corresponding connection A_4, curvature F_4, and Dirac operator ∂_4.

Consider the curvature action $\int_{S^8} F_8 \wedge {}^*F_8$. Note that F_8 is a 2-form and ${}^*F_8$ is a 6-form. Using the splittings $\text{spin}(8) = H + \text{spin}(8)/H$ and $S^8 = S^4 + S^8/S^4$, split F_8 into components in H along S^4 (denoted by F_{8H}) and components in $\text{spin}(8)/H = S^3 \times S^1$ along $S^8/S^4 \approx S^3 \times \mathbb{R}P^1$ (denoted by $F_{8G/H}$), so that $F_8 = F_{8H} + F_{8G/H}$. Then the action can be written as

$$\int_{S^8} (F_{8H} \wedge {}^*F_{8H} + F_{8H} \wedge {}^*F_{8G/H} + F_{8G/H} \wedge {}^*F_{8H} + F_{8G/H} \wedge {}^*F_{8G/H})$$

The term $\int_{S_8} F_{8H} \wedge {}^*F_{8H}$, after integration over $E/M = S^3 \times \mathbb{R}P^1$, becomes a pure Yang–Mills action term for gauge group H over base manifold $M = S^4$, or, in effect, $\int_{S^4} F_4 \wedge {}^*F_4$, where F_4 is a 2-form in the generalized Lie algebra $H = S^2 \times S^4 \times S^4 \times S^6 \times SU(3)$.

The terms

$$\int_{S^8} (F_{8H} \wedge {}^*F_{8G/H} + F_{8G/H} \wedge {}^*F_{8H})$$

become

$$\int_{S^4} D_M \Phi(G/H) \wedge {}^*D_M \Phi(G/H)$$

where D_M is the covariant gradient in the directions of $M = S^4$, and where $\Phi(G/H)$ is the spin(8) Clifford algebra 0-form corresponding to the invariance of A_8 with respect to the vector field in G/H along the E/M direction of $E = S^8$.

According to Mayer (1981), the term $\int_{S^8} F_{8G/H} \wedge {}^*F_{8G/H}$ becomes

$$\int_{S^4} \{[\Phi(G/H), \Phi(G/H)] - \Phi([G/H, G/H])\}^2$$

Therefore the total action over S^4 becomes

$$\int_{S^4} F_4 \wedge {}^*F_4 + \int_{S^4} (D_M\Phi(G/H) \wedge {}^*D_M\Phi(G/H) + \{[\Phi(G/H), \Phi(G/H)] - \Phi([G/H, G/H])\}^2)$$

The term

$$\int_{S^4} (D_M\Phi(G/H) \wedge {}^*D_M\Phi(G/H) + ([\Phi(G/H), \Phi(G/H)] - \Phi([G/H)] - \Phi([G/H, G/H])\}^2)$$

is just the standard Ginzburg–Landau–Higgs potential corresponding to four Higgs scalars arising from the four-dimensional spaces E/M and G/H that give mass by spontaneous symmetry breaking to the three weak vector bosons of the massive weak boson $SU(2)$ part of the spin(4) component of spin(8). The spin(8) Higgs mechanism is similar to that described in Section 22.3 of Lee (1981) for an $SO(3)$ gauge group, except that the two modes of a zero-mass neutral weak boson and the one mode of a massive Higgs scalar described by Lee are combined in the spin(8) theory into three modes of a massive neutral weak boson, denoted by W^0.

The term $\int_{S^4} F_4 \wedge {}^*F_4$ is a pure Yang–Mills action term for gauge group

$$H \approx S^2 \times S^4 \times S^4 \times G_2 = S^2 \times S^4 \times (S^4 \times S^6) \times SU(3)$$

over base manifold $M = S^4$. When the massive W^0 resulting from the Higgs mechanism is added, the gauge bosons of spin(8) gauge field theory that act over base manifold S^4 correspond to

$$\mathbb{R}P^1 \times H \approx \mathbb{R}P^1 \times S^2 \times S^4 \times (S^4 \times S^6) \times SU(3)$$
$$\approx S^3 \times S^4 \times (S^4 \times S^6) \times SU(3)$$

Now consider the spinor matter action $\int_{S^8} \langle \varphi, \partial_8 \varphi \rangle$. As suggested by Mayer (1981), begin with the spinors on S^8 with group spin(8). In Section 6, generalized spinor matter fields were constructed whose fibres are $Q^8 = Q^8_+ \oplus Q^8_-$, with $Q^8_\pm = S^7 \times \mathbb{R}P^1$ and $Q^8 = S^7 \times S^1$. Therefore Q^8 corresponds locally to the octonions $\mathbb{O}$ by the local correspondences of S^7 with the imaginary octonions and S^1 with the real octonions. Globally, the decomposition $Q^8 = Q^8_+ \oplus Q^8_-$ corresponds to the twofold covering of $\mathbb{R}P^1$ by S^1.

Let ∂_8 be the generalized Dirac operator for spin(8) Yang–Mills over S^8. The operator ∂_8 is the first-order differential operator whose symbol is Clifford multiplication (Parker, 1982). The connection A_8 defines a covariant derivative ∇_8, which maps the spinor space sections $\Gamma(Q^8_\pm)$ to $\Gamma(Q^8_\pm \otimes T^*S^8)$. Since, for spin(8) gauge field theory, the base manifold S^8 is considered to be equivalent to $S^8 - \{\infty\} = \mathbb{R}^8$, the base manifold dual tangent space $T^*S^8 \approx T^*\mathbb{R}^8$ corresponds to the vector space of the spin(8) Clifford algebra. Clifford multiplication by a vector element in $T^*\mathbb{R}^8$ interchanges the even and odd subspaces of the spin(8) Clifford algebra, and therefore interchanges the even and odd half-spinor spaces Q^8_+ and Q^8_-, so that Clifford multiplication σ: $Q^8_\pm \otimes T^*\mathbb{R}^8 \to Q^8_\mp$. The operator ∂_8 is defined by

$$\partial_8 \colon \Gamma(Q^8_\pm) - \nabla_8 \to \Gamma(Q^8_\pm \otimes T^*\mathbb{R}^8) - \sigma \to \Gamma(Q^8_\mp)$$

Let ∂_4 be the corresponding generalized Dirac operator for $H = S^2 \times S^4 \times S^4 \times G_2$ Yang–Mills–Higgs over S^4 with the same Q^8 spinor matter field fibres. Then, if φ is a spinor matter field, the spinor matter action terms should be $\int_{S^4} \langle \varphi, \partial_4 \varphi \rangle + \langle \varphi, \Phi\varphi \rangle$ (a Yakawa term).

Therefore, the total action for spin(8) gauge field theory over S^4 should be of the form

$$\int_{S^4} F_4 \wedge {}^*F_4 + (D_M\Phi \wedge {}^*D_M\Phi + ([\Phi, \Phi] - \Phi)^2) + \langle \varphi, \partial_4\varphi \rangle + \langle \varphi, \Phi\varphi \rangle$$

Some interesting relationships involving the spinor matter term may prove useful in further analysis.

Note that G_2, the automorphism group of the octonions $\mathbb{O}$ and therefore the local automorphism group of Q^8, appears in spin(8) $= S^7 \times S^7 \times G_2$ along with two copies of the intersection of two hemispheres of S^8.

Since the two-component open cover transition function of S^8 for spin(8) Yang–Mills takes values in spin(8) $= S^7 \times S^7 \times G_2$, it naturally corresponds to both a map from the boundary of one open covering set to the boundary of the other ($S^7 \times S^7$) and to a local automorphism of the associated spinor matter fibre Q^8 (G_2). Therefore the spinor matter fibres Q^8 are naturally related to the duality structure of spin(8) Yang–Mills over S^8 that Gürsey and Tze (1983) have compared to the duality structure of spin(4) $= S^3 \times S^3$ Yang–Mills over S^4. As observed by Grossman et al. (1984), the last Hopf map shows that spin(8) is the standard Yang–Mills gauge group over $S^8 = \mathbb{O}P^1$ with instanton number 1.

For the $\mathbb{R}P^1 \times H$ Yang–Mills over S^4, $\mathbb{R}P^1 \times H \approx S^3 \times S^4 \times S^4 \times G_2$ naturally corresponds to a map between the two copies of the entire base manifold S^4 related to electromagnetism and the translational part of de

Sitter gravitation, to the G_2 local automorphism of the associated spinor matter fibre Q^8 related to the color force and the Lorentz part of gravitation, which carries color charge, and to S^3 related to the $SU(2)$ weak bosons.

11. FORCE STRENGTHS

The relative strengths of the force of gravitation, the color force, the weak force, and the force of electromagnetism should be determined in part by their relationships to the base manifold S^4 and each (half) spinor manifold $Q^8_\pm$ of spin(8) gauge field theory.

The other part of their relative strengths should be determined by considering them to be proportional to the ratio of the square of the electron mass to the square of the characteristic mass, if any, associated with the forces. The electron mass m_e is the only mass term that is not calculable in spin(8) gauge field theory, in which it is a fundamental quantity like the speed of light c and Planck's constant $\hbar$.

The characteristic mass term only applies to gravitation, for which the mass term is the ratio of the square of the electron mass to the square of the Planck mass, and to the weak force, for which the mass term is the ratio of the square of the electron mass to the sum of the squares of the weak vector boson masses. Neither the electromagnetic force nor the color force has massive gauge bosons or Planck-mass-type characteristic masses.

This section deals with the part of the relative strengths of the force of gravitation, the color force, the weak force, and the force of electromagnetism due to their relationships to the base manifold S^4 and each (half) spinor manifold $Q^8_\pm$ of spin(8) gauge field theory, called the geometric part of the relative strengths.

First, consider the base manifold S^4. Each of the four forces has a natural global action on a part of S^4:

Gravitation has gauge group spin(5), which has a natural global action on spin(5)/spin(4) $= S^4$. Therefore the base manifold component of the geometric part of the strength of gravitation is S^4.

The color force has gauge group $SU(3)$, which has a natural global action on

$$SU(3)/S(U(2)\times U(1)) = \mathbb{C}P^2$$

$\mathbb{C}P^2$ is the same as S^4 except for structure at ∞. Therefore the base manifold component of the geometric part of the strength of the color force is S^4.

The weak force has gauge group spin(4), but that is reducible by spin(4) $= SU(2)\times SU(2)$ and spontaneous symmetry breaking to $SU(2)$, which has a natural global action on $SU(2)/U(1) = S^2$. S^4 contains S^2. The base manifold component of the geometric part of the strength of one-half of the weak force is S^2.

Electromagnetism has gauge group $U(1)^4$, the maximal torus of spin(8), but that is reducible to $U(1)$ by considering each of the four $U(1)$'s in $U(1)^4$ to be one space-time component of the photon in the path-integral formulation of quantum electrodynamics (Leighton, 1959). $U(1)$ has a natural global action on $U(1)=S^1$. S^4 contains S^1. The base manifold component of the geometric part of the strength of one-fourth of electromagnetism is S^1.

Second, consider the (half) spinor manifold $Q^8_\pm$. Each of the four forces has a natural local action on $Q^8_\pm$, which extends to the full $Q^8_\pm$ because $Q^8_\pm$ is parallelizable. To determine which part of $Q^8_\pm$ corresponds to each of the four forces, it is useful to recall that $Q^8_\pm = S^7 \times \mathbb{R}P^1$ is the Silov boundary of an irreducible symmetric bounded domain of type IV_8, denoted by

$$D^8_\pm \approx \mathrm{spin}(10)/\mathrm{spin}(8) \times U(1)$$

Gravitation has gauge group spin(5), so that it has a natural local action on an irreducible symmetric bounded domain of type IV_5,

$$D^5_\pm \approx \mathrm{spin}(7)/\mathrm{spin}(5) \times U(1)$$

$D^5_\pm$ has Silov boundary $Q^5_\pm = S^4 \times \mathbb{R}P^1$. $Q^8_\pm$ contains $Q^5_\pm$. The (half) spinor component of the geometric part of the strength of gravitation is $Q^5_\pm = S^4 \times \mathbb{R}P^1$.

The color force has gauge group $SU(3)$, so that it has a natural local action on an irreducible symmetric bounded domain of type $I_{1,3}$,

$$D^{1,3}_\pm \approx SU(4)/S(U(3) \times U(1)) \approx B^6$$

$D^{1,3}_\pm$ has Silov boundary $Q^{1,3}_\pm = S^5$. $Q^8_\pm$ contains $Q^{1,3}_\pm$. The (half) spinor component of the geometric part of the strength of the color force is $Q^{1,3}_\pm = S^5$.

Each of the $SU(2)$ gauge groups in the spin(4) of the weak force has a natural local action on an irreducible symmetric bounded domain of type IV_3,

$$D^3_\pm \approx \mathrm{spin}(5)/SU(2) \times U(1)$$

$D^3_\pm$ has Silov boundary $Q^3_\pm = S^2 \times \mathbb{R}P^1$. $Q^8_\pm$ contains $Q^3_\pm$. The (half) spinor component of the geometric part of the strength of one-half of the weak force is $Q^3_\pm = S^2 \times \mathbb{R}P^1$.

Each of the $U(1)$ gauge groups in the $U(1)^4$ of electromagnetism has a natural local action on $U(1)=S^1$. $Q^8_\pm$ contains S^1. The (half) spinor component of the geometric part of the strength of one-fourth of electromagnetism is S^1.

Third, note that in some cases the dimension of the base manifold component differs from the dimension of the (half) spinor component. For the full spin(8) theory over S^8 prior to dimensional reduction, the (half) spinor space $Q^8_\pm$ and the base manifold S^8 are both eight-dimensional, a fact that is related to the isomorphism, peculiar to spin(8), of spinor and vector representations. Therefore, in Yang–Mills theory for spin(8) over S^8, the dual tangent space $T^*(S^8)$ of the base manifold can act on the (half) spinor space $Q^8_\pm$ by a Clifford multiplication map $T^*(S^8)\otimes Q^8_\pm \to Q^8_\mp$ that is the symbol of a generalized Dirac operator ∂_8: $\Gamma(Q^8_\pm)\to\Gamma(Q^8_\mp)$ (Parker, 1982). Such a structure is necessary for a gauge field theory to have spinor matter fields that are compatible with the gauge fields over the base manifold.

Therefore in case of forces whose base manifold component and (half) spinor component have different dimension, it is necessary to include a dimensional adjustment factor.

Gravitation has a five-dimensional (half) spinor component $Q^5_\pm = S^4\times \mathbb{R}P^1$, which is the Silov boundary of $D^5_\pm$. The base manifold component S^4 is four-dimensional. The dual tangent space $T^*(S^4)$ of the base manifold is generated by the unit S^4 pseudoscalar function $I^4(x)$, $x\in S^4$, of the tangent Clifford algebra of S^4 as defined in Chapter 4 of Hestenes and Sobczyk (1984). For $I^4(x)$ to act by Clifford multiplication on $Q^5_\pm$, the dimensionality of $Q^5_\pm$ must be reduced from five to four and the volume $V(Q^5_\pm)$ must be divided by a length L_5. To calculate L_5, recall that $Q^5_\pm$ is the Silov boundary of $D^5_\pm$. If the volume $V(D^5_\pm)$ is put into a four-dimensional cube, such as $I^4(x)$, then the gravitational dimensional adjustment factor L_5 is the fourth root of the volume $V(D^5_\pm)$ of $D^5_\pm$.

The dimensional adjustment factor L_5 is the least intuitively clear part of spin(8) gauge field theory. A similar factor was used by Wyler (1971) in calculating his value of the electromagnetic fine structure constant, and the difficulty in finding a physical interpretation for it was a major factor in criticisms of Wyler's work (Gilmore, 1972).

The color force has a five-dimensional (half) spinor component $Q^{1,3}_\pm = S^5$, which is the Silov boundary of $D^{1,3}_\pm$. The base manifold component S^4 is four-dimensional. The color force dimensional adjustment factor $L_{1,3}$ should be the fourth root of the volume $V(D^{1,3}_\pm)$ of $D^{1,3}_\pm$.

One-half of the weak force has a three-dimensional (half) spinor component $Q^3_\pm = S^2\times\mathbb{R}P^1$, which is the Silov boundary of $D^3_\pm$. The base manifold component S^2 is two-dimensional. The dimensional adjustment factor for one-half of the weak force should be the square root of the volume $V(D^3_\pm)$ of $D^3_\pm$.

The electromagnetism (half) spinor component and the base manifold are both S^1, so no dimensional adjustment factor is needed.

Fourth, and finally, the geometric part of the strength of each of the four forces is calculated, using volumes from Hua (1963), as follows:

Gravitation has a geometric strength of the volume $V(S^4) = 8\pi^2/3$ of the base manifold component, multiplied by the volume $V(Q^5_\pm) = 8\pi^3/3$ of the spinor component, and divided by the fourth root of the volume $V(D^5_\pm)=(\pi^5/2^4 5!)^{1/4}$. Therefore the geometric strength of gravitation is $V_G=2^8\pi^4(15/2\pi)^{1/4}/9=3444.0924$.

The color force has a geometric strength of the volume $V(S^4)=8\pi^2/3$ of the base manifold component, multiplied by the volume $V(Q^{1,3}_\pm)=4\pi^3$ of the spinor component, and divided by the fourth root of the volume $V(D^{1,3}_\pm=(\pi^3/6)^{1/4}$. Therefore the geometic strength of the color force is $V_C=2^5\pi^4(6\pi)^{1/4}/3=2164.978$.

One half of the weak force has a geometric strength of the volume $V(S^2)=4\pi$ of the base manifold component, multiplied by the volume $V(Q^3_\pm)=4\pi^2$ of the spinor component, and divided by the square root of the volume $V(D^3_\pm)=(\pi^3/24)^{1/2}$. Therefore the geometric strength of one-half of the weak force is $V_W=2^5\pi^2(6/\pi)^{1/2}=436.46599$.

One fourth of electromagnetism should have a geometric strength that is the volume $V(S^1)=2\pi$ of the base manifold component, which is also the volume of the spinor component. The Abelian gauge force of electromagnetism does not have any strength reduction factor. Therefore the geometric strength of one-fourth of electromagnetism is $V_E=2\pi=6.2831853$.

Therefore, mass factors for the weak force and gravitation aside, the relative strengths of the forces are:

Gravitation:	$\alpha_G=V_G/V_G=1$
Color force:	$\alpha_C=V_C/V_G=0.6286062$
Weak force:	$\alpha_W=2V_W/V_G=0.2534577$
Electromagnetism:	$\alpha_E=4V_E/V_G=1/137.03608$

12. LATTICE GAUGE STRUCTURE AND QUANTIZATION

If the base manifold for spin(8) gauge field theory (either S^8 before dimensional reduction or S^4 after dimensional reduction) is continuous, then there are no nearest neighbor relationships for the spinor spaces Q^8 at any point on the base manifold, because any path between two base manifold points contains infinitely many intervening points. For a path integral formulation of physics, nearest neighbor relationships are useful to explain the transition between fermions at two nearby base manifold points linked by gauge bosons.

To get such nearest neighbor relationships, a lattice should be used as base "manifold," or base lattice. Regular lattice structures in four and eight dimensions are described in Coxeter (1973). Lattices on n-dimensional spheres correspond to $(n+1)$-dimensional polytopes. There are only three regular $(n+1)$-dimensional regular polytopes (generalized cube, octahedron, and tetrahedron) for $n = 4$ and for $n = 8$, so there is no arbitrarily fine regular global lattice structure for S^4 or S^8. Therefore, to get a regular lattice structure that can be made arbitrarily fine for lattice gauge theory regularization to approximate base manifolds S^4 or S^8, the lattice structure must be local.

For $S^8 = \mathbb{O}P^1$, the natural lattice structure is based on Gossett's eight-dimensional honeycomb 5_{21} that represents the integral octonions (Coxeter, 1973). The vertices of 5_{21} in $\mathbb{R}^8$ have as coordinates all sets of eight integers, or eight halves of odd integers with an even sum.

For $S^4 = \mathbb{H}P^1$, the natural lattice structure is based on the honeycomb $\{3, 3, 4, 3\}$ that represents the integral quaternions (Coxeter, 1973). The vertices of $\{3, 3, 4, 3\}$ in $\mathbb{R}^4$ have as coordinates all sets of four integers, or four halves of odd integers.

The lattice structure can cover at most $S^n - \{\infty\}$. Therefore the smallest number of neighborhoods that can cover S^n with lattice structure is two, one covering the "south pole" at 0 and the other covering the "north pole" at ∞, with boundary conditions at the overlap of the two neighborhoods at the equator. This may correspond to the necessity in quantum theory to have at least two things, the observer and the observed. The twofold lattice covering of S^n corresponds to the construction of the transition function M used in dimensional reduction of the spin(8) base manifold from S^8 to S^4, following the construction of Problem 3 of Chapter Vbis of Choquet-Bruhat et al. (1982).

The four-dimensional base lattice is locally $\{3, 3, 4, 3\}$, with spin(8) gauge bosons at the links of the lattice and with spin(8) acting through its gauge bosons on elementary fermions at the vertices.

Spin(8) gauge field theory should be quantized by a lattice gauge quantization procedure similar to that described in Creutz (1983). Creutz (1983) uses a hypercubic lattice with a Wilson action for square plaquettes. The plaquettes are two-dimensional square faces of the lattice hypercubes, and there are six plaquettes per vertex in the hypercubic lattice (Coxeter, 1973).

For the $\{3, 3, 4, 3\}$ lattice of spin(8) gauge field theory, plaquettes would be two-dimensional triangular faces, and there are 32 plaquettes per lattice vertex (Coxeter, 1973). As a general vertex in a $\{3, 3, 4, 3\}$ lattice can be considered to be the center of a hypercube, 32 of the 96 plaquettes at a lattice vertex can be thought of as corresponding to the central vertex.

As it is the lattice of integral quaternions, $\{3, 3, 4, 3\}$ is naturally related to Lorentz invariance. A general vertex can be considered to be the center of a hypercube (16 vertices) and the centers of the neighboring hypercubes (eight more vertices), or equivalently, to be the center of a 24-cell (24 vertices). If any of the 24 neighbor vertices is chosen to be the future timelike direction, then the opposite neighbor vertex is the past timelike direction, the eight neighbor vertices closest to the future neighbor vertex form the future light-cone, the eight neighbor vertices closest to the past neighbor vertex form the past light-cone, and the six remaining neighbor vertices define the three positive and three negative space directions.

Note that a choice of future timelike direction at any vertex in the $\{3, 3, 4, 3\}$ lattice determines space-time structure for the entire lattice.

13. WEAK BOSON MASSES

In a lattice gauge theory, the gauge bosons are on the links and the fermions are on the vertices. The spin(4) weak force is reduced by spontaneous symmetry breaking to spin(3) $= SU(2)$, as indicated by the self-duality of the bivectors of the $1+4+6+4+1=16$-dimensional Clifford algebra $C4$ for spin(4). The symmetry-breaking mechanism may be described by considering two lattice links connected by a common vertex. Consider the first link as carrying a massless gauge boson corresponding to any of the six infinitesimal generators of spin(4). Spontaneous symmetry breaking should require that the gauge boson carried by the second link be such that the net result of the two links taken together should be the same as one of the three infinitesimal generators of $SU(2)$, which should be identified with the W^+, W^-, and W^0. The notation W^0 is used here to distinguish this pure weak symmetry breaking, with no ad hoc Higgs bosons and no direct involvement of electromagnetism, from conventional electroweak theory.

Mass production by "natural" symmetry breaking, without artificially introducing Higgs fields, was done by Finkelstein et al. (1963), who used quaternionic field theory to construct two charged massive vector fields (analogous to the W^+ and W^-) and one neutral massless vector field (analogous to the photon). The technique used in Section 10 is similar to that of Mayer (1981).

The further decomposition of the three weak vector bosons into the neutral W^0 and the charged pair W^+ and W^- is related to the decomposition of S^3 into $S^1 \times S^2$ by the Hopf fibration $S^1 \to S^3 \to S^2$.

In spin(8) gauge theory, the masses of the first-generation W bosons should come from the stable first-generation particles and antiparticles associated with the vertex joining the two links. The sum of the masses of the stable first-generation fermion particles and antiparticles has been calculated as 3.754 GeV. The sum of the masses of the $W1^+$, $W1^-$, and $W1^0$ should be 3.754 GeV multiplied by the ratio of the weak force strength to the electromagnetic force strength:

$$m_{W1^+,W1^-,W1^0} = 2(V_W/2V_E)3.754\ \text{GeV} = 260.774\ \text{GeV}$$

The factor V_E is multiplied by 2 because there are two $U(1)$'s for each $SU(2)$'s in spin(4). The other factor of 2 is for the two helicity states of each massive fermion.

To determine the first-generation masses of $W1^+$, $W1^-$, and $W1^0$ individually, consider that $SU(2)$ can be identified with the unit quaternions S^3; that S^3 has a Hopf fibration $S^1 \to S^3 \to S^2$; and that S^2 should correspond to $W1^+$ and $W1^-$ while S^1 should correspond to $W1^0$.

The unit sphere S^3 in R^4 contains the point (1/2, 1/2, 1/2, 1/2); the corresponding point S^2 is $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$; and the corresponding point in S^1 is $(1/\sqrt{2}, 1/\sqrt{2})$. Then the ratio of the sum of the $W1^+$ and $W1^-$ masses to the $W1^0$ mass should be

$$(2/\sqrt{3})\,V(S^2)/(2/\sqrt{2})\,V(S^1) = 1.632993$$

Therefore

$$m_{W10} = 260.774/(1+1.632993) = 99.04\ \text{GeV}$$

As the masses of the $W1^+$ and $W1^-$ should be equal, $m_{W1^+} = m_{W1^-} =$ 80.97 GeV.

Therefore the observed first-generation weak force constant G_{W1} should be given by

$$G_{W1} = \alpha_W m_e^2/(m_{W1^+}^2 + m_{W1^-}^2 + m_{W1^0}^2) = 2.886 \times 10^{-12}$$

and $G_{W1}m_{\text{proton}}^2 = 0.97 \times 10^{-5}$, where the proton mass is calculated from the constituent masses of the up and down quarks to be 938.2626 MeV, or 1836.118 times the electron mass.

Similar calculations can be done using the second- and third-generation fermion particles and antiparticles. For the second generation, $W2^{\pm}$ has a mass of about 329 GeV and $W2^0$ has a mass of about 403 GeV. For the third generation, $W3^{\pm}$ has a mass of about 17,549 GeV and $W3^0$ has a mass of about 21,492 GeV. The corresponding observed weak force constants should be smaller than the first-generation G_{W1} by factors of about $(1/4.0711)^2 \approx 1/16.57$ for the second generation and about $(1/217.004)^2 \approx 1/47{,}091$ for the third generation. The sum for all three generations of $G_W m_{\text{proton}}^2$ is about 1.03×10^{-5}.

14. KOBAYASHI–MASKAWA THEORY

The three generations of W bosons may be related to the Kobayashi–Maskawa (1973) theory of generation mixing of quarks, particularly as parametrized by Chau and Keung (1984). The mixing matrix between the up, charm, and truth quarks and the down, strange, and bottom quarks is due to the action on them of the weak force, and can be written as follows:

	d	s	b
u	V_{ud}	V_{us}	V_{ub}
c	V_{cd}	V_{cs}	V_{cb}
t	V_{td}	V_{ts}	V_{tb}

In the Chau–Keung parametrization, the matrix (V_{ij}) is given by the product of three 3×3 matrices:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & cy & sy \\ 0 & -sy & cy \end{pmatrix}\begin{pmatrix} cz & 0 & sz\,e^{-i\Phi} \\ 0 & 1 & 0 \\ -sz\,e^{i\Phi} & 0 & cz \end{pmatrix}\begin{pmatrix} cx & sx & 0 \\ -sx & cx & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The first matrix, involving the angle y in the terms $cy=\cos y$ and $sy=\sin y$, gives the generation mixing between the third and second generations. I conjecture that $sy=m_{W2}/(m_{W2}^2+m_{W3}^2)^{1/2}=0.0187571$.

The second matrix, involving the angle z in the terms $cz=\cos z$ and $sz=\sin z$, gives the generation mixing between the third and first generations. I conjecture that $sz=m_{W1}/(m_{W1}^2+m_{W3}^2)^{1/2}=0.00460816$.

The second matrix also involves the phase angle Φ. I assume, perhaps arbitrarily, that skipping the second genertion introduces a 90° phase angle, so that $\Phi=90°$.

The third matrix, involving the angle x in the terms $cx=\cos x$ and $sx=\sin x$, gives the generation mixing between the second and first generations. I conjecture that $sx=m_{W1}/(m_{W1}^2+m_{W2}^2)^{1/2}=0.2385428$.

Using the conjectured values for the parameters gives the following Kobayashi–Maskawa matrix:

	d	s	b
u	0.9710	0.239	0
c	-0.239 $-0.00008i$	0.9708 $-0.00002i$	0.019
t	0.0045 $-0.0045i$	-0.0018 $-0.001i$	0.9998

The calculated KM matrix is close to currently accepted values, except that the calculated $|V_{ts}|\approx|V_{cb}|\approx 0.019$. The currently accepted value, based on a truth quark mass $m_t \approx 45$ GeV, is about 0.05 (Chau and Keung, 1984). What would be the effect if $m_t \approx 130$ GeV?

Note that the conventional KM parametrization is related to the Chau–Keung parametrization approximately as follows (Chau and Keung, 1984) (see Figure 2):

$$sx \approx s_1 \approx |V_{us}|$$

$$sz \approx s_1 s_3 \approx |V_{ub}|$$

$$s\Phi \approx s_2 s\delta / sy$$

$$sy \approx (s_2^2 + s_3^2 + 2 s_2 s_3 c\delta)^{1/2} \approx |V_{cb}|$$

For the spin(8) values of $sx \approx 0.239$, $sz \approx 0.239$, $sz \approx 0.0046$, $\Phi \approx 90°$, and $sy \approx 0.019$, the corresponding values of the conventional parameters are $s_1 \approx 0.239$, $s_3 \approx 0.019$, $\delta \approx 135°$, and $s_2 \approx 0.027$.

Inami and Lim (1981) have examined the effect on KM theory of $m_t >$ weak boson mass.

From Sections 3.1.a. and 3.2.a. of Inami and Lim (1981),

$$|(c_1 s_2 + c_2 t_3 c_\delta) s_2| \, |\bar{C}(x_t, 0)| \approx K \leq 0.9 \times 10^{-2}$$

where $t_3 = \tan\theta_3 \approx s_3$ and $x_t = m_t / m_{W^\pm}$.

Use equation (2.14) of Inami and Lim (1981)

$$\bar{C}(x_t, 0) = \tfrac{3}{4}[x_t/(x_t - 1)]^2 \ln(x_t) + \tfrac{1}{4} x_t - \tfrac{3}{4} x_t/(x_t - 1)$$

to define $C(m_t) = |\bar{C}(x_t, 0)|$.

Therefore, as K is constant, and as $c_1 \approx 0.97$ and $c_2 \approx 1$ are substantially independent of m_t,

$$0.97 s_2(m_t)^2 + s_3(m_t) c_\delta(m_t) s_2(m_t) = K / C(m_t)$$

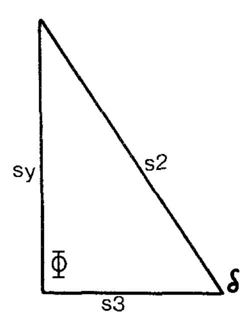


Fig. 2.

To apply the Inami–Lim techniques to compare the spin(8) theoretical values with experimental results, first calculate that

$$C(45\text{ GeV}) \approx 0.24, \qquad C(130\text{ GeV}) \approx 1.33$$

Then use the spin(8) parameter values and $C(130\text{ GeV}) \approx 1.33$ to calculate K, getting $K \approx 0.00059$.

Therefore, for $m_t \approx 45$ GeV,

$$0.97 s_2^2 + s_3 c_\delta s_2 \approx 0.0025$$

Also, from the relationships between parameters and the value of $|V_{cb}| \approx 0.05$ for $m_t \approx 45$ GeV,

$$s_2^2 + 2 s_3 c_\delta s_2 \approx 0.0025 - s_3^2$$

Comparing the last two equations shows that the spin(8) values of the KM parameters at $m_t \approx 130$ GeV give a value of the constant K that is consistent with current experimental values for $|V_{cb}|$ interpreted for $m_t \approx 45$ GeV, provided that, for $m_t \approx 45$ GeV, $c_\delta \approx -s_3/s_2$. As that value of c_δ is within the currently accepted range, the spin(8) theoretical values of KM parameters are consistent with existing experimental results.

15. QCD AND PION MASS

As pointed out by Isgur and Karl (1983), the quarks and gluons of the $SU(3)$ color force are roughly analogous to the leptons and photon of the $U(1)^4$ electromagnetic force, so that crude estimates can be made of color force phenomena.

Hadrons that are observed directly are considered to be composite particles made up of quarks bound together primarily by the color force gluons. The observed hadrons can be accounted for by requiring that within a given composite hadron, all quarks of a given flavor (d, u, s, c, b, or t) have parallel spin, and all antiquarks of a given flavor within the same composite hadron have parallel spin. The composite hadrons (in which quarks are bound by color force gluons) can be considered to be roughly analogous to atoms (in which leptons are bound by electromagnetic force photons).

It might also be thought that since gluons carry color charge, there should be bound states of gluons, called glueballs. However, as stated by Moriyasu (1983), quoting Coleman (1977) and Coleman and Smart (1977), a classical glueball cannot exist because if each gluon were attracted to all other neighboring gluons, then the gluons would be antiparallel to each other, thus contradicting the continuity of the classical gluon field. From the lattice gauge field point of view of spin(8) gauge field theory, the requirement of continuity of the classical gluon field should be replaced by the requirement that gluons (like quarks of the same flavor) should be oriented parallel to each other within the characteristic radius (typical hadron radius) of the color force.

Note that neither the electromagnetic force strength nor the color force strength involves mass terms, so that the electromagnetic fine structure constant $\alpha_E = 1/137.03608$ and the color constant $\alpha_C = 0.6286062$ should give the strengths of the respective forces at an "equilibrium" distance that can be crudely estimated by the respective Bohr radii for "atoms" with first-generation leptons or quarks.

For electromagnetism (Lee, 1981), the Bohr radius

$$BR_E \approx 1/\alpha_E m_e \approx 137 \times 4 \times 10^{-11}\ \text{cm} \approx 5 \times 10^{-9}\ \text{cm}$$

which is just the Bohr radius of the hydrogen atom.

For the color force, the Bohr radius

$$BR_C \approx 1/\alpha_C m_d \approx 1/\alpha_C m_u \approx 4 \times 10^{-11}\ \text{cm}/(0.6286 \times 612) \approx 1 \times 10^{-13}\ \text{cm}$$

which is approximately the radius of a proton. As the pion, a first-generation quark-antiquark pair, is the primary carrier of the strong force that results from the color force, it is reasonable to expect that the effective pion mass, or the square root of the squares of the masses of the neutral and charged pions, should be roughly given by the mass whose Compton wavelength is the radius of the proton, or, equivalently, the color force Bohr radius. Therefore

$$(m_\pi^2 + m_{\pi\pm}^2)^{1/2} \approx 1/BR_C \approx 10^{13}\ \text{cm}^{-1} \approx 400 m_e \approx 200\ \text{MeV}$$

As the experimental value of the pion mass is about 135 MeV for the neutral pion and about 139.6 MeV for the charged pions, the experimental value is about $(135^2 + 139.6^2)^{1/2} \approx 194.2$ MeV.

The 200 MeV value also roughly agrees with the characteristic length $l \approx (200 \pm 50\ \text{MeV})^{-1}$ of QCD renormalization theory as given in Equation (23.124) of Lee (1981). For high energy, and with three generations of quarks, Lee (1981) gives $\alpha_C = 6\pi/[21 \ln(l \cdot EQ)]$, where EQ is the energy of a "typical" quark mass. Lee (1981) uses $\alpha_C \approx 0.39$, and gets $EQ \approx 1.6$ GeV. For spin(8) theory, $\alpha_C \approx 0.6$, so that $EQ \approx 900$ MeV. It seems that either set of values is reasonably consistent with experiment, considering the very crude estimation techniques used.

An interesting conjecture as to the pion mass is based on an observation of P. Stanbury (personal communication, 1985) that is roughly equivalent to the equation $BE_\pi \approx (2/3)m_d + 2m_\pi$, where $BE_\pi = 2m_d - m_\pi$ is the binding energy of a pion. Stanbury's observation implies that $3m_\pi \approx (4/3)m_d$, so that $m_\pi \approx (4/9)m_d \approx 139$ MeV. A reasonable physical conjecture for the basis of the equation $BE_\pi \approx (2/3)m_d + 2m_\pi$ might be that the $(2/3)m_d$ term comes from the "cancellation" of one-third of each of the two constituent quarks of the pion, and that the $2m_\pi$ term comes from "sharing" a virtual pion–antipion pair.

In any event, given the pion mass and the long-range color force constant (which may be derivable from α_C), Guralnick et al. (1984) have shown that lattice gauge theory QCD calculations on a Cray XMP have given reasonable results for meson and baryon masses and strong-force couplings of π and ρ mesons.

16. PLANCK MASS

To estimate the Planck mass, use the lattice gauge structure of spin(8) gauge field theory to estimate the mass of a one-vertex universe, as that should correspond to the Planck mass.

Note that the only stable massive elementary fermions are those of the first generation, as second- and third-generation massive leptons and quarks decay into massless μ-neutrinos or τ-neutrinos and massive first-generation leptons and/or quarks.

Consider the Planck mass particle to be the sum over all possible combinations of particle–antiparticle pairs of first-generation fermions at one vertex. Since a one-vertex lattice has no links, there are no gauge bosons to carry away any of the pairs. There are eight fermionparticles and eight fermion antiparticles, for a total of 64 particle–antiparticle pairs. A typical combination should have several quarks, several antiquarks, a few colorless quark–antiquark pairs that would be equivalent to pions, and some leptons and antileptons.

Consider the contribution of independent fermion leptons and quarks to the mass sum. The neutrino is massless, and the electron and positron each have mass of only about 1/2000 GeV. The up and down quarks and antiquarks each have mass of almost 1/3 GeV. As there are three colors of the color force, there are 12 distinct such quarks and antiquarks. Due to the Pauli exclusion principle, no fermion lepton or quark could be present at the vertex more than twice. Due to the color force requirement that all quarks of a given flavor and all antiquarks of a given flavor within a given composite particle (i.e., within the color force characteristic radius of $\sim 10^{-13}$ cm) should have parallel spin, no quark or antiquark could be present at the vertex more than once.

Therefore the total contribution to the Planck mass of independent elementary fermions is only $\sim 12/3$ GeV $= 4$ GeV.

Pions, colorless quark–antiquark pairs, are bosons and are not subject to the Pauli exclusion principle. Of the 64 particle–antiparticle pairs, 12 are pions, each having mass of about 0.14 GeV. A typical combination should have about six pions. If all the pions are independent, the typical combination should have a mass of 0.14×6 GeV $= 0.84$ GeV. However, just as the pion mass of 0.14 GeV is less than the sum of the masses of a quark and an antiquark, pairs of oppositely charged pions may form a bound state of less mass than the sum of two pion masses. If such a bound state of oppositely charged pions has a mass as small as 0.1 GeV, and if the typical combination has one such pair and four other pions, then the typical combination should have a mass in the range of 0.66 GeV.

Summing over all 2^{64} combinations, the total mass of a one-vertex universe should give $m_{\text{Planck}} \approx (1.217\text{–}1.550) \times 10^{19}$ GeV. However, the method of estimating is so rough that it is probably fairer to say that the estimated value should be roughly $m_{\text{Planck}} \approx (1\text{–}1.6) \times 10^{19}$ GeV.

Therefore the observed gravitational constant G_G should be given by

$$G_G = \alpha_G (m_e^2 / m_{\text{Planck}}^2) \approx (1\text{–}2.6) \times 10^{-45}$$

and $G_G m_{\text{proton}}^2 \approx (3.4\text{–}8.8) \times 10^{-39}$.

17. DE SITTER GRAVITATION

MacDowell and Mansouri (1977) have formulated gravitation as a gauge theory. They showed that a spin(5) = sp(2) Yang–Mills gauge group over a four-dimensional base manifold such as S^4 produces de Sitter gravitation, that is, Einstein gravitation plus a cosmological term.

De Sitter gravitation with gauge group spin(5) = sp(2) as described by MacDowell and Mansouri (1977) may form a basis for cosmological models such as that of Gott (1982) in which bubbles of Minkowski space are formed by quantum tunneling in a de Sitter false vacuum with Hawking radiation. The Minkowski space undergoes an early exponential expansion phase, causing particle creation that in turn causes a phase transition to a standard Robertson–Walker $k = -1$ open universe like ours.

Gravitation in spin(8) gauge field theory has ten gravitons (eight charged and two neutral). The eight charged gravitons should be confined by a mechanism similar to the gluon confinement mechanism. Graviton confinement should be to within the characteristic radius of the gravitational force, about the Planck length, $\sim 10^{-33}$ cm.

Bound states of charged gravitons should not exist, in analogy with the argument of Moriyasu (1983), quoting Coleman (1977) and Coleman and Smart (1977), against the existence of classical color charge glueballs, because gluons within the characteristic color force radius should be oriented parallel to each other.

18. COMPARISON WITH EXPERIMENT

The experimental values in Table V are taken from various sources, and may be rounded off or approximate. The spin(8) values may be rounded off from the values calculated in this paper. Evidently, there is pretty good (but not perfect) agreement with all the known values except the value for the truth quark.

CERN has announced that the truth quark has been observed to have a mass of about 40 GeV (Rubbia, 1984). The observation is based on decay of the W^+ boson into a meson made up of a truth quark and a bottom antiquark, as well as the corresponding decay of the W^- boson into a meson made up of a truth antiquark and a bottom quark. Such decay should produce two jets, a charged lepton, and a neutrino. CERN identifies the

Table V.[a]

Quantity	Spin(8)	Experiment
$m_{e\text{-neutrino}}$	0	~0
m_d, MeV	312.8	≈350
m_u, MeV	312.8	≈350
m_μ, MeV	104.8	105.7
$m_{\mu\text{-neutrino}}$	0	~0
m_s, MeV	523	≈550
m_c, GeV	1.99	≈1.7
m_τ, GeV	1.88	1.78
$m_{\tau\text{-neutrino}}$	0	~0
m_b, GeV	5.63	≈5.2
m_t, GeV	130	?
$m_{W1^+} = m_{W1^-}$, GeV	81	~81
m_{W1^0}, GeV	99	~93
$m_{W2^+} = m_{W2^-}$, GeV	329	?
m_{W20}, GeV	403	?
$m_{W3^+} = m_{W3^-}$, TeV	17.5	?
m_{W30}, TeV	21.5	?
m_{Planck}, GeV	$\sim(1\text{-}1.6)\times 10^{19}$	1.22×10^{19}
α_E	1/137.03608	1/137.03604
$G_W m^2_{\text{proton}}$	1.03×10^{-5}	1.02×10^{-5}
α_C	0.6286	~1
$G_G m^2_{\text{proton}}$	$\sim(3.4\text{-}8.8)\times 10^{-39}$	5.9×10^{-39}
Kobayashi-Maskawa-Chau-Keung parameters:		
Cabibbo angle, sin x	0.239	0.23
sin y	0.0188	0.019
sin z	0.0046	0.005
Φ, deg	90(?)	?

[a] In spin(8) gauge field theory the speed of light, Planck's constant, and the electron mass are given and everything else is calculated from them.

truth quark or antiquark as one jet, the charged lepton, and the neutrino, which are observed to have a total mass of about 40 GeV.

If the CERN mass for the truth quark is correct, then spin(8) gauge field theory is wrong, because it predicts a truth quark mass of about 130 GeV.

Could the 40-GeV jet, lepton, and neutrino in W^+ and W^- decay be something other than a truth quark or antiquark? CERN has also observed W^0 decay that produces an electron-positron pair with energy of about 50 GeV plus a hard photon (Lubkin, 1983). If the truth quark mass were about 40 GeV, then W^0 decay into a meson made up of a truth quark and a truth antiquark would produce an 80-GeV electron-positron pair, not the observed 50-GeV electron-positron pair, not the observed 50-GeV pair.

If the 40-GeV jet, lepton, and neutrino in W^+ and W^- decay do not correspond to a meson with a truth quark or antiquark, but rather to a decay scheme similar to the W^0 decay producing an electron-positron pair at 50 GeV, then spin(8) gauge field theory might be correct.

Such a decay scheme could come from a tendency of a W boson to decay into two roughly equal parts. In spin(8) lattice gauge field theory any fermion is on a vertex and any boson, except W, is on a link. However, a W boson is on two links connected by a vertex. When a W boson decays, its mass-energy splits into two parts corresponding to the two links. The theory gives a mass of about 80 GeV for W^+ and W^- and a mass of about 100 GeV for W^0, so many W^+ and W^- decay products should be around 40 GeV and many W^0 decay products should be around 50 GeV. The theory is consistent with observations of hard photons and multiple jets.

CERN has observed enhanced production of jets with an energy of about 147 GeV (Waldrop, 1984). That energy is roughly consistent with the existence of a particle containing a truth quark having the mass predicted by spin(8) gauge field theory, about 130 GeV. A truth quark with mass of about 130 GeV might also account for observations of single jets that, with associated neutral particles, have a total energy of about 120 GeV and observations of an electron, neutrino, and as many as three jets all having total energy as much as 160 GeV.

One way to decide the matter experimentally might be to look at the energy region 250–300 GeV to see if truth-antitruth mesons are produced, as would be expected from spin(8) gauge field theory.

Another relevant type of experiment uses the Kobayashi–Maskawa theory (Kobayashi and Maskawa, 1973; Lee, 1981) of relationships among fermions of different generations and observations of CP-violating phenomena. The relationship between Kobayshi–Maskawa theory and spin(8) gauge field theory has been discussed in the section on Kobayashi–Maskawa theory.

In addition to the problems of fitting the CERN 45-GeV truth quark mass with standard Kobayashi–Maskawa theory, as of the summer of 1985 CERN was having difficulty confirming its interpretation of the 45-GeV events. According to Miller (1985), the UA1 experimenters had observed many events clustering near the $W^{\pm}$ mass rather than distributed as would be expected by the CERN truth quark model, and the UA2 experimenters had still not seen any convincing event for a 45-GeV truth quark.

I think that truth is the proper name for the t-quark, whose phenomena can confirm or refute the predictions of spin(8) gauge field theory.

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